LIE THEORY: THE TOPOLOGY OF GROUPS

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Hand-out: abhimanyu.io/media/wu-urop.pdf

More on Lie theory: thewindingnumber.blogspot.com/p/1203.html
More on Topology: thewindingnumber.blogspot.com/p/2204.html

STUFF TO COVER

- Motivation
- Part I: the theory
  - Infinitesimal generators, exponential map
  - Lie bracket, adjoint map, Jacobi identity
  - Lie algebra elements as derivations
  - BCH formula, Lie's theorems, covering spaces
  - Examples of the Lie correspondence
- Part II: abstraction
- Applications
  - Topology of Indefinite Orthogonal Group
  - Parallel parking and control theory

MOTIVATION

"Set theory : Topology :: Group theory : Lie theory"

MOTIVATION

- $(ab)^{1/2} \neq a^{1/2}b^{1/2}$ vs. $(a+b)/2 \ [\mathrm{mod}\, n] \neq a/2 + b/2 \ [\mathrm{mod}\, n]$
- But circle group not cyclic
- Idea
- “Spatial structure”

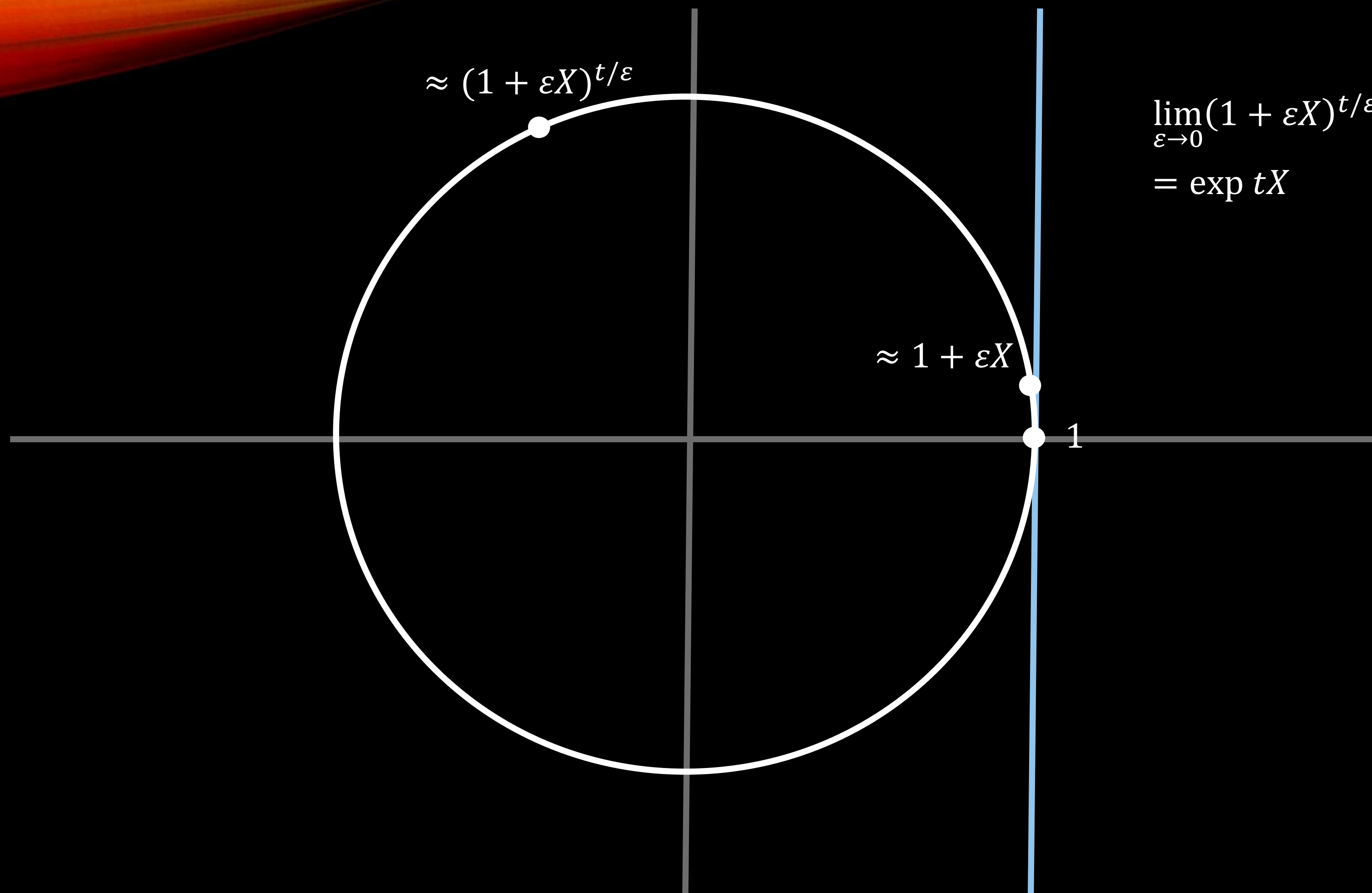
MOTIVATION

Cardinality in set theory "generalizes" to "wiggle room" in topology

- $[0,10]_{\mathbb{R}}$ is "kinda" finite, it's "like" $[0,10]_{\mathbb{Z}}$

In group theory, we're not just concerned with cardinality, but group structure

- $(\mathbb{R},+)$ is "like" $(\mathbb{Z},+)$
- $U(1)$ is "kinda" cyclic, "kinda" finite
- Generators, dimension, etc.



$$\lim_{\varepsilon \to 0}(1+\varepsilon X)^{t/\varepsilon}$$
$$= \exp tX$$

Equivalently intuition for:

- Euler's formula
- Compound-interest limit
- Infinitesimal generators

ALTERNATE FORM OF EXP

Expand $\left(1+\frac{X}{n}\right)^n$

k^{th} degree term is $\frac{1}{n^k}\binom{n}{k}X^k$

$$\lim_{n\to\infty}\frac{1}{k!}\frac{n(n-1)\ldots(n-k+1)}{n^k}=\frac{1}{k!}$$

$$\Rightarrow \exp X = 1 + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots$$

EXP

- In general, infinitesimal generators form a vector space TG

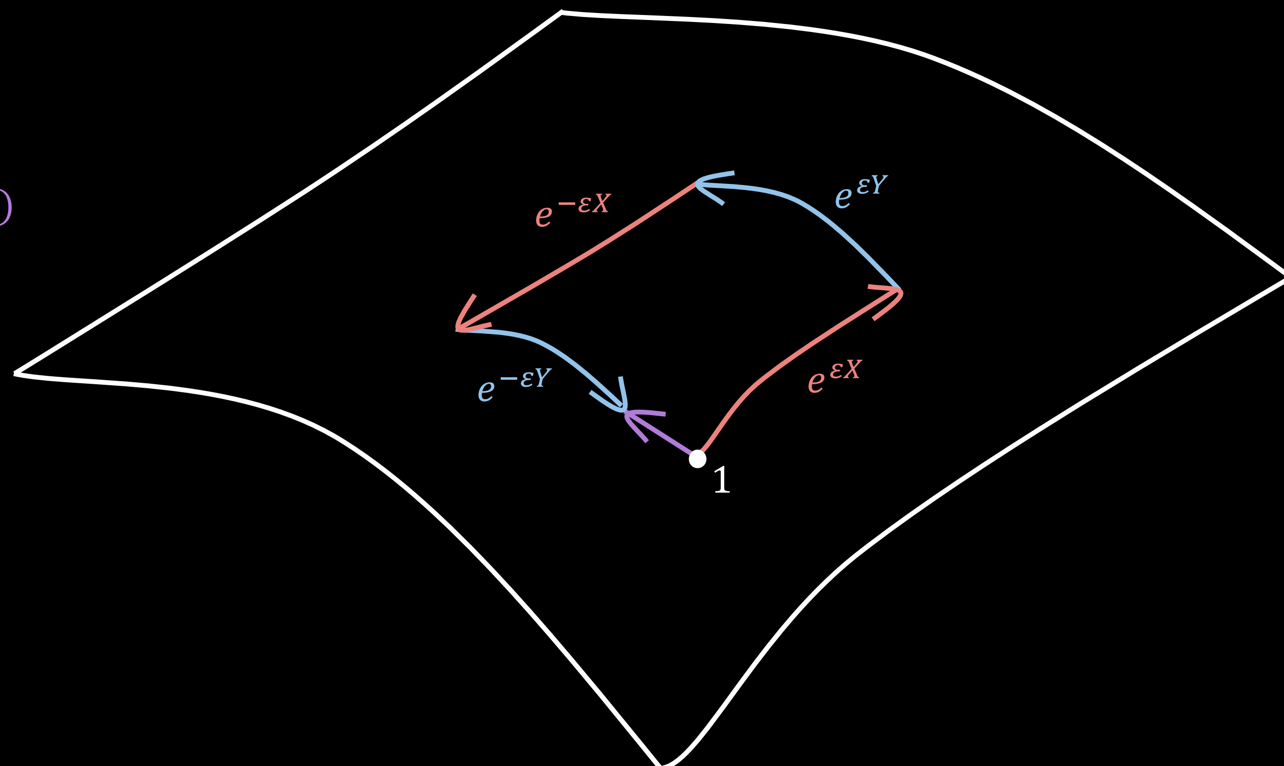
$$\exp: TG \to G$$

- Clearly for an Abelian group, $\exp(X + Y) = \exp X \, \exp Y$ and the tangent space is homomorphic to the group

- Analog of FT-FAG: *every compact connected Abelian Lie group is a torus*

- Interesting local structure of the group comes from non-Abelian stuff

$$e^{\varepsilon X}e^{\varepsilon Y}e^{-\varepsilon X}e^{-\varepsilon}$$
$$\approx 1+\varepsilon^2(XY-YX)$$



ANOTHER WAY TO SEE [*X*,*Y*]

- "Adjoint map" homomorphism
- Induces a Lie algebra homomorphism
- Preserves Lie bracket – *Jacobi identity*

$$\mathrm{Ad} : G \to \mathrm{Aut}\,(G) := g \mapsto \lambda x.\, gxg^{-1}$$

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}\,(\mathfrak{g}) := Y \mapsto [Y, -]$$

$$\begin{aligned} & \mathrm{ad}([X,Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)] \\ \Rightarrow\ & \mathrm{ad}([X,Y])(Z) = \mathrm{ad}(X)\mathrm{ad}(Y)Z - \mathrm{ad}(Y)\mathrm{ad}(X)Z \\ \Rightarrow\ & [[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]] \end{aligned}$$

TAYLOR SERIES

$$\exp \nabla = \Delta$$

DERIVATIONS

- Cayley's theorem: groups can be understood as the automorphism groups of some object $G = \mathrm{Aut}(M)$
- Derivatives of automorphisms at the identity are directional derivative operators, or "derivations"

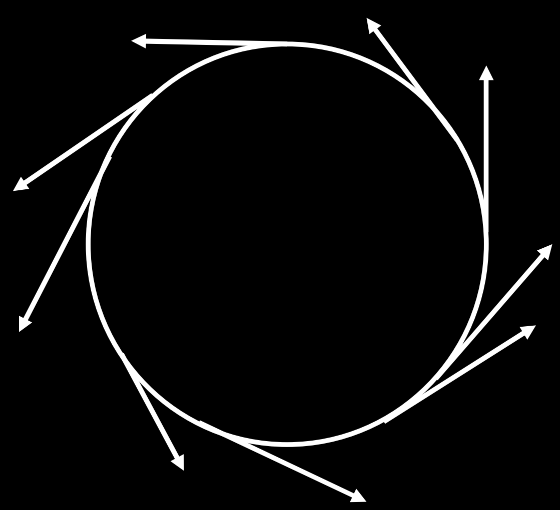
$$\phi_p(fg) = \phi_p(f)\phi_p(g)$$

$$fg + \varepsilon d\phi_0(fg) = (f + \varepsilon d\phi_0 f)(g + \varepsilon d\phi_0 g)$$

$$\Rightarrow d\phi_0(fg) = (d\phi_0 f)g + f(d\phi_0 g)$$

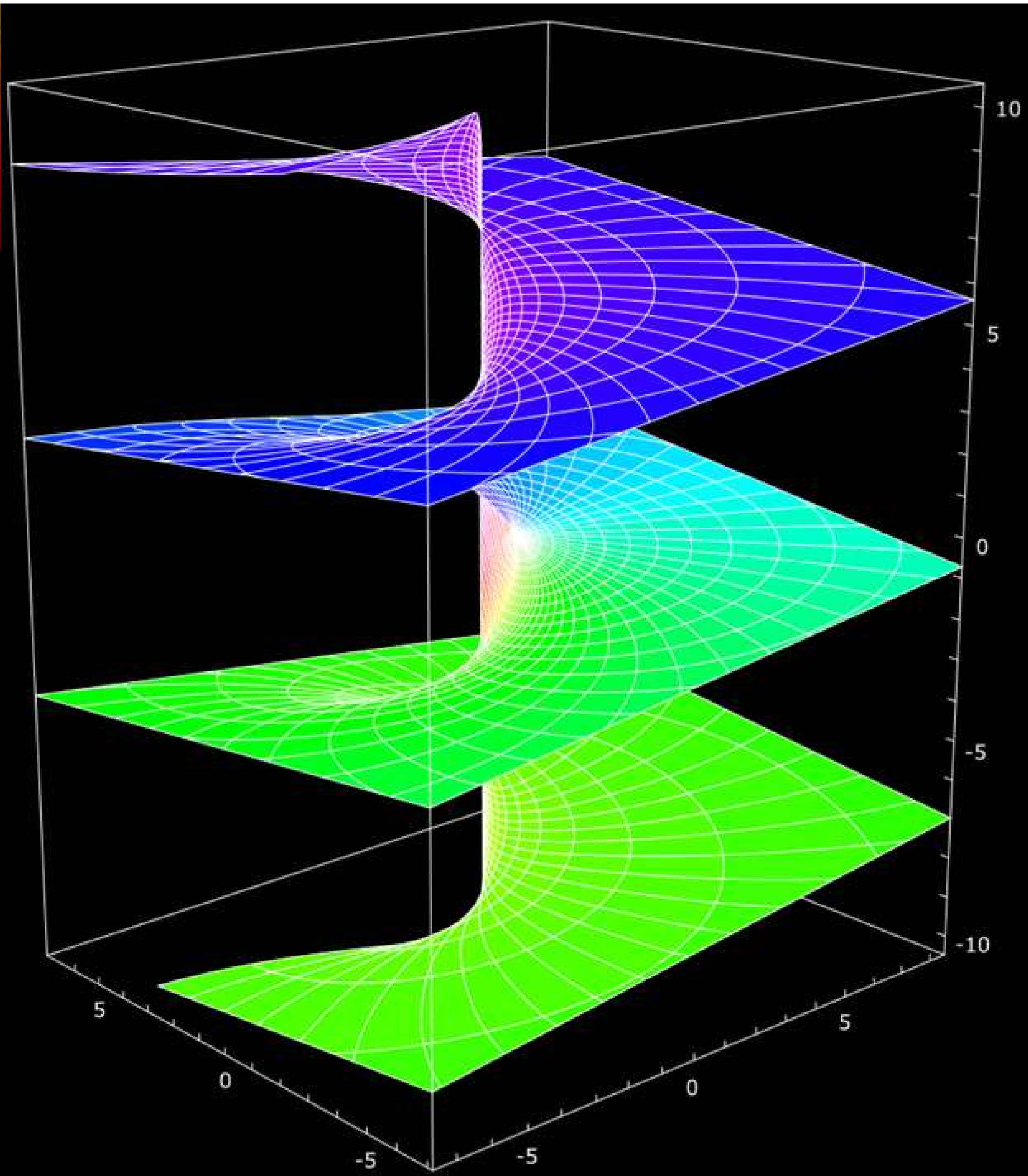
DERIVATIONS

- Tangent vectors ⇒ Left-invariant vector fields ⇒ Directional derivatives
- "Linear operator that satisfies product rule" is equivalent to being a directional derivative for analytic functions



HOW MUCH STRUCTURE IS PRESERVED LOCALLY?

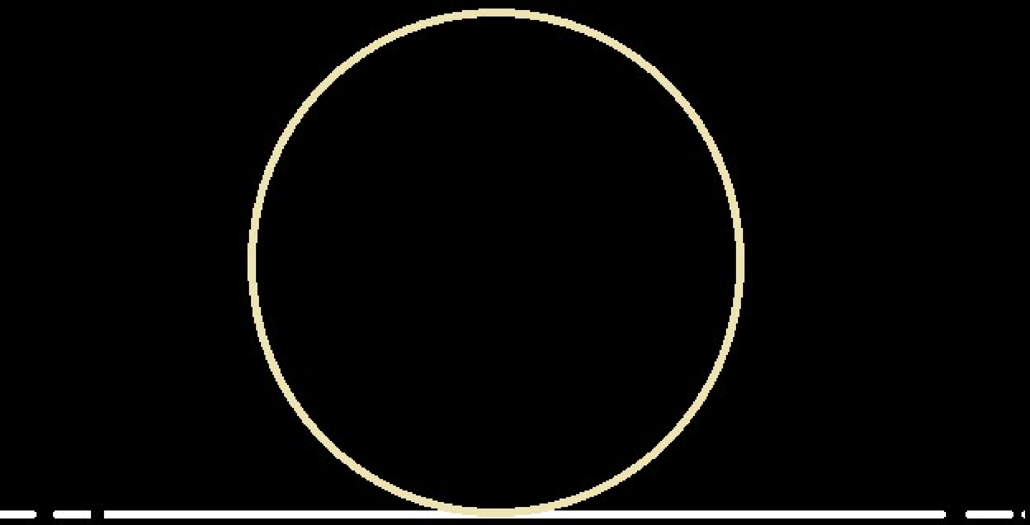
- Sufficiently high-order structure surely determines local structure of group: exp is a local topological isomorphism
- An example of local topological isomorphism is having the same covering space: for connected groups, this is the *only* local Lie group isomorphism



COVERING SPACE (EXAMPLE)

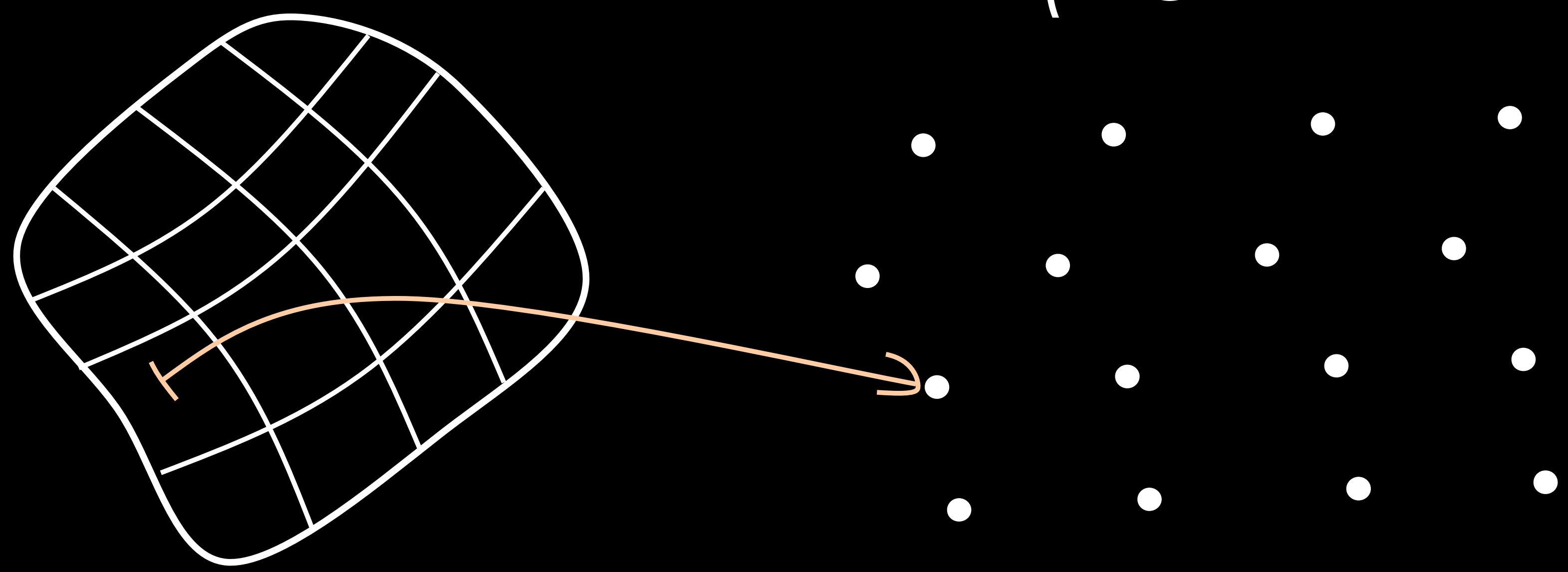
Source:
en.wikipedia.org/wiki/File:
Riemann_surface_log.svg

COVERING SPACE (EXAMPLE)



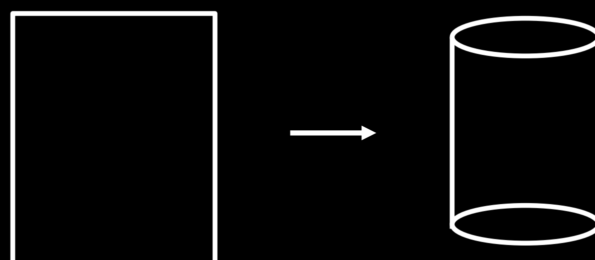
More generally, any plane covers a torus

COVERING SPACE (NON-EXAMPLE)



UNIVERSAL COVER IS SIMPLY-CONNECTED

- Notion of "wrapping": must be a homeomorphism on some open neighbourhood of each point from each connected pre-image
- Any "wrapping" must introduce circles, i.e. eliminate simply-connectedness



- A universal cover is "unwrapping"

DEFINITION OF A LIE ALGEBRA

- Definitions (motivation in Lie's second, third theorems)
  - *Lie algebra* – a vector space equipped with an antisymmetric, bilinear, Jacobi-satisfying product called the "Lie bracket".
  - *Lie algebra homomorphism* – a linear, Lie bracket-preserving map.

- Motivation comes from the *2nd and 3rd fundamental theorems of Lie theory*.

- Definition isn't obvious nonsense: *1st fundamental theorem of Lie theory*.

LIE CORRESPONDENCE

- **Lie's first theorem**
  - Tangent space of a Lie group is a Lie algebra
  - Differential of a Lie group homomorphism is a Lie algebra homomorphism

- **Lie's second theorem –** (G simply-connected) a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ is the differential of a unique Lie group homomorphism $G \to H$

- **Lie's third theorem –** Every Lie algebra is the tangent space of a unique simply-connected Lie group

BAKER-CAMPBELL-HAUSDORFF

$\log(e^X e^Y)$ can be written purely in terms of nested commutators of X and Y

(proof: abhimanyu.io/media/wu-urop.pdf p. 5-6)

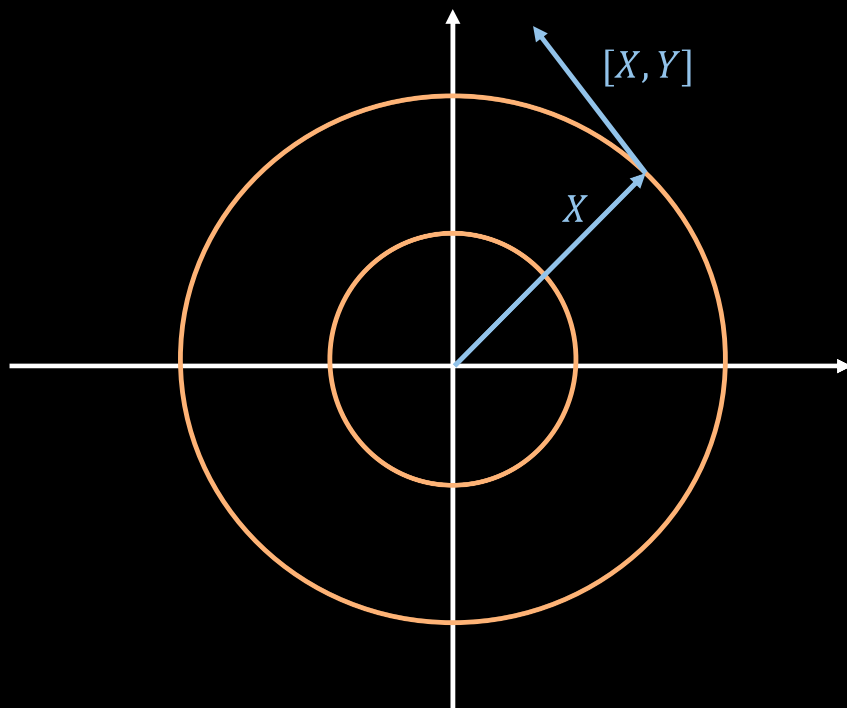
EXAMPLES OF THE CORRESPONDENCE

Lie group : Lie algebra

- **Subgroup : Subalgebra**
- **Normal subgroup : Ideal**
- **Centre : Centre**
- **Automorphisms : Derivations**

KILLING FORM

- Contours of Killing form are orbits of Adjoint map
- Radius perpendicular to tangent: generalisation of $x \cdot (x \times y) = 0$



ABSTRACTION

ABSTRACT LIE THEORY

- *Topological group*: group that is also topological space
- *Lie group*: both group that is also smooth manifold
- *Lie algebra*: a vector space with a cross product satisfying the axioms

- *Lie algebra to a Lie group*: tangent space, equipped with Lie bracket given by $\frac{1}{\varepsilon^2}(\Phi_X^\varepsilon \Phi_Y^\varepsilon - \Phi_Y^\varepsilon \Phi_X^\varepsilon)$. Similarly define adjoint map.

- Given tangent vector X, exists a unique one-parameter subgroup $x\colon \mathbb{R} \to G$ such that $x'(0) = X$ and $x(s+t) = x(s)x(t)$. Let $\exp X = x(1)$.

REPRESENTATION THEORY

- Lie group representation: linear action of a Lie group on a vector space
- Equivalently: Lie group homomorphism $G \to \mathrm{GL}(V)$

- Lie algebra representation: Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$

- Example: adjoint representation

- Correspondence between Lie group and algebra representations (follows from Lie's theorems)

REPRESENTATION THEORY

- Can all Lie groups be seen as matrix Lie groups?
- No.
- Ado’s theorem: every finite-dimensional Lie algebra has a faithful matrix representation.

INDEFINITE ORTHOGONAL GROUP

physics.stackexchange.com/a/36425

PARALLEL PARKING

- $\text{Steer} = \frac{\partial}{\partial \theta}$
- $\text{Drive} = \sin(\theta + \phi)\frac{\partial}{\partial x} - \cos(\theta + \phi)\frac{\partial}{\partial y} - \cos\theta\frac{\partial}{\partial \phi}$

- $\left[[\text{Steer}, \text{Drive}], \text{Drive}\right] = \sin\phi\frac{\partial}{\partial x} - \cos\phi\frac{\partial}{\partial y}$