A PRIMER ON LIE THEORY

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Hand-out: abhimanyu.io/media/wu-urop.pdf

More on Lie theory: thewindingnumber.blogspot.com/p/1203.html
More on Topology: thewindingnumber.blogspot.com/p/2204.html

STUFF TO COVER

- Motivation
- Infinitesimal generators, exponential map
- Lie bracket, adjoint map, Jacobi identity
- Lie algebra elements as derivations
- BCH formula, Lie's theorems, covering spaces
- Examples of the Lie correspondence
- Killing form, simplicity
- Applications: indefinite orthogonal group
- Applications: control theory (parallel parking)

MOTIVATION

9:0: information: print result

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@[_refl_lemma]
theorem FUNDAMENTAL_THEOREM_OF_MATHEMATICS :
set_theory / topology = group_theory / lie_theory :=
rfl
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“Se ”

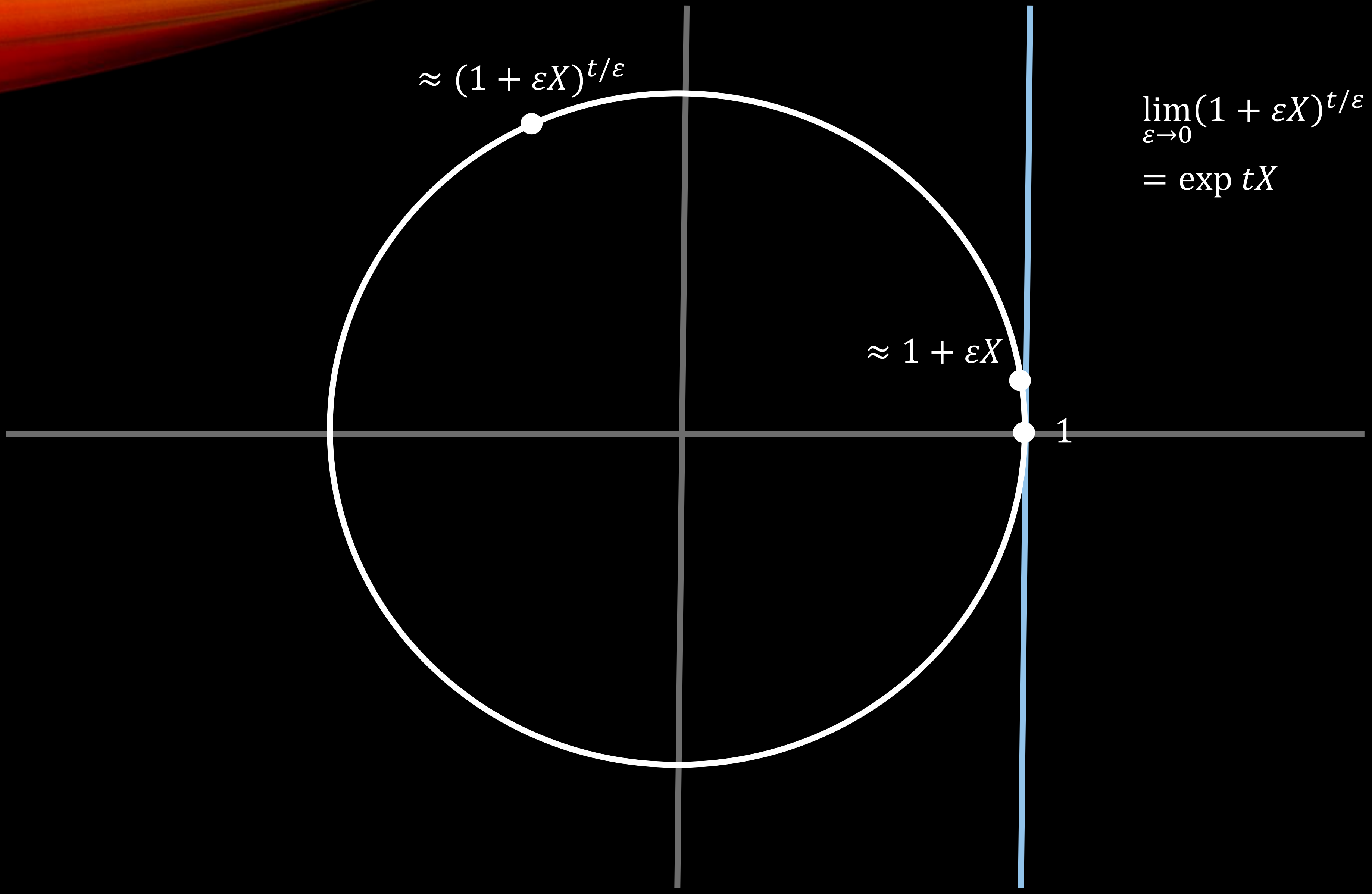
MOTIVATION

Cardinality in set theory “generalizes” to “wiggle room” in topology

- $[0,10]_{\mathbb{R}}$ is “kinda” finite, it’s “like” $[0,10]_{\mathbb{Z}}$

In group theory, we’re not just concerned with cardinality, but group structure

- $(\mathbb{R},+)$ is “like” $(\mathbb{Z},+)$
- $U(1)$ is “kinda” cyclic, “kinda” finite
- $(\mathbb{R},+)$ and $(\mathbb{R}^2,+)$ are “different”



$$\lim_{\varepsilon \to 0} (1+\varepsilon X)^{t/\varepsilon}$$
$$= \exp tX$$

Equivalently intuition for:

- Euler’s formula
- Compound-interest limit
- Infinitesimal generators

ALTERNATE FORM OF EXP

Expand $\left(1+\frac{X}{n}\right)^n$

k^{th} degree term is $\frac{1}{n^k}\binom{n}{k}X^k$

$$\lim_{n\to\infty}\frac{1}{k!}\frac{n(n-1)\ldots(n-k+1)}{n^k}=\frac{1}{k!}$$

$$\Rightarrow \exp X = 1 + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots$$

EXP

- In general, infinitesimal generators form a vector space TG

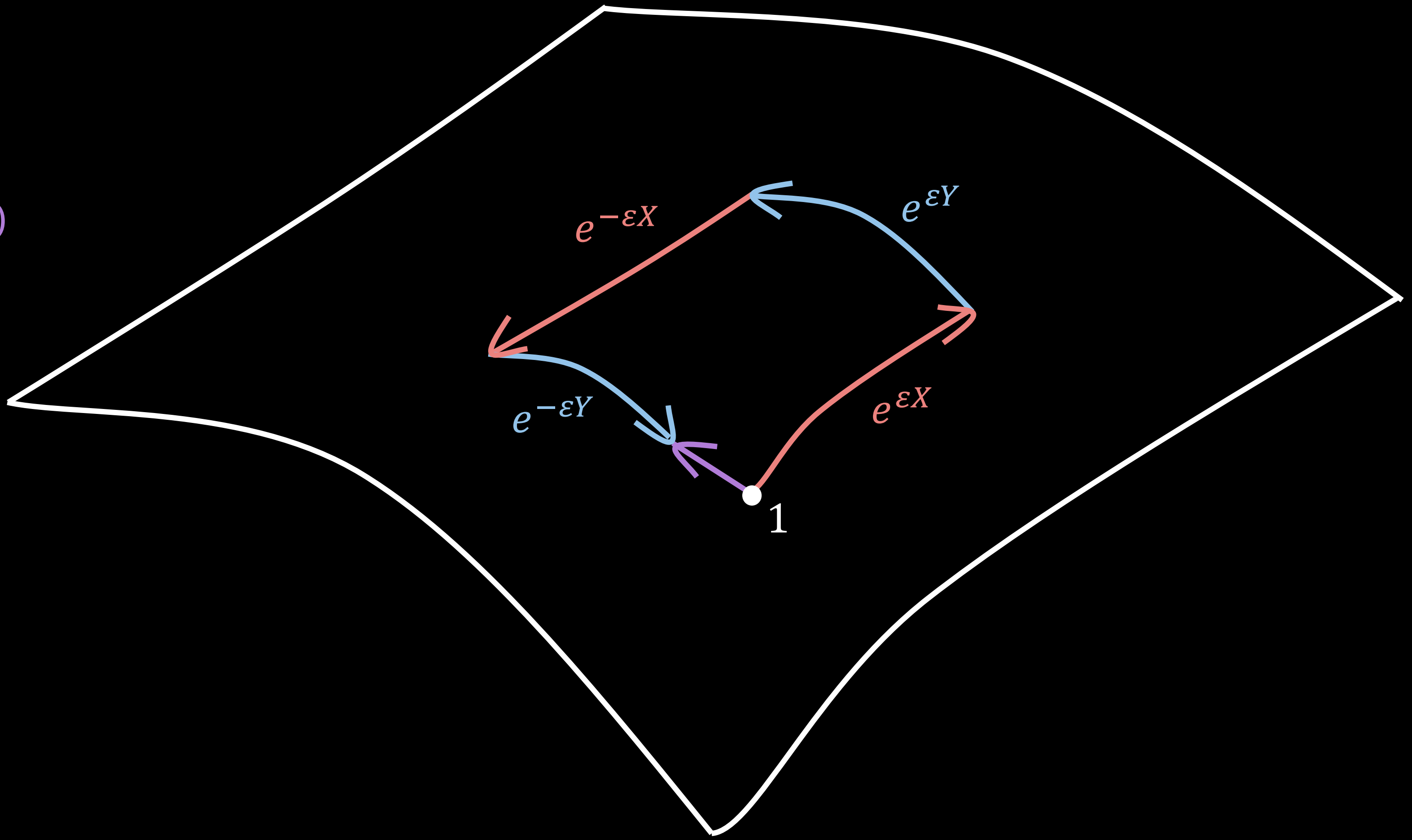
$$\exp: TG \to G$$

- Clearly for an Abelian group, $\exp(X+Y) = \exp X \exp Y$ and the tangent space is homomorphic to the group

- Analog of FT-FAG: *every compact connected Abelian Lie group is a torus*

- Interesting local structure of the group comes from non-Abelian stuff

$$e^{\varepsilon X}e^{\varepsilon Y}e^{-\varepsilon X}e^{-\varepsilon Y}$$
$$\approx 1 + \varepsilon^2(XY - YX)$$



ANOTHER WAY TO SEE [*X*,*Y*]

- "Adjoint map" homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}\,(G) := g \mapsto \lambda x.\ gxg^{-1}$
- Induces a Lie algebra homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}\,(\mathfrak{g}) := Y \mapsto [Y, -]$
- Preserves Lie bracket – *Jacobi identity*

$$\begin{aligned} \mathrm{ad}([X,Y]) &= [\mathrm{ad}(X), \mathrm{ad}(Y)] \\ \Rightarrow\ \mathrm{ad}([X,Y])(Z) &= \mathrm{ad}(X)\mathrm{ad}(Y)Z - \mathrm{ad}(Y)\mathrm{ad}(X)Z \\ \Rightarrow\ [[X,Y],Z] &= [X,[Y,Z]] - [Y,[X,Z]] \end{aligned}$$

TAYLOR SERIES

$$\exp \nabla = \Delta$$

DERIVATIONS

- Cayley's theorem: groups can be understood as the automorphism groups of some object $G = \mathrm{Aut}(M)$
- Derivatives of automorphisms at the identity are directional derivative operators, or "derivations"

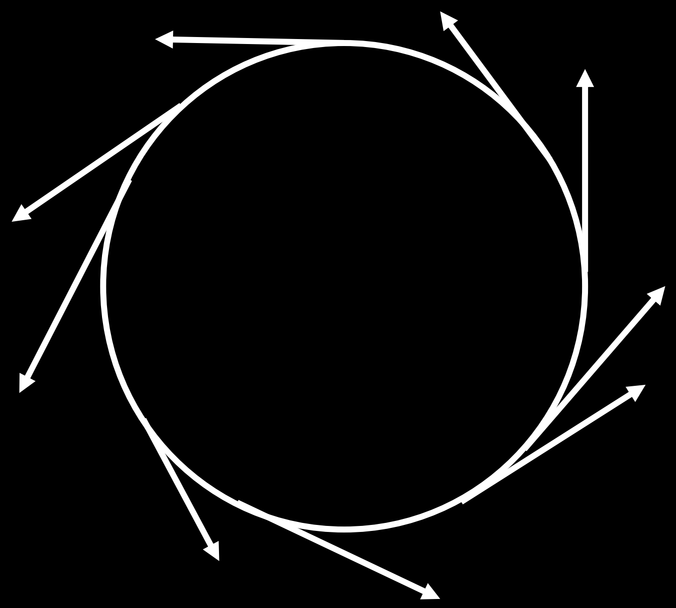
$$\phi_p(fg) = \phi_p(f)\phi_p(g)$$

$$fg + \varepsilon d\phi_0(fg) = (f + \varepsilon d\phi_0 f)(g + \varepsilon d\phi_0 g)$$

$$\Rightarrow d\phi_0(fg) = (d\phi_0 f)g + f(d\phi_0 g)$$

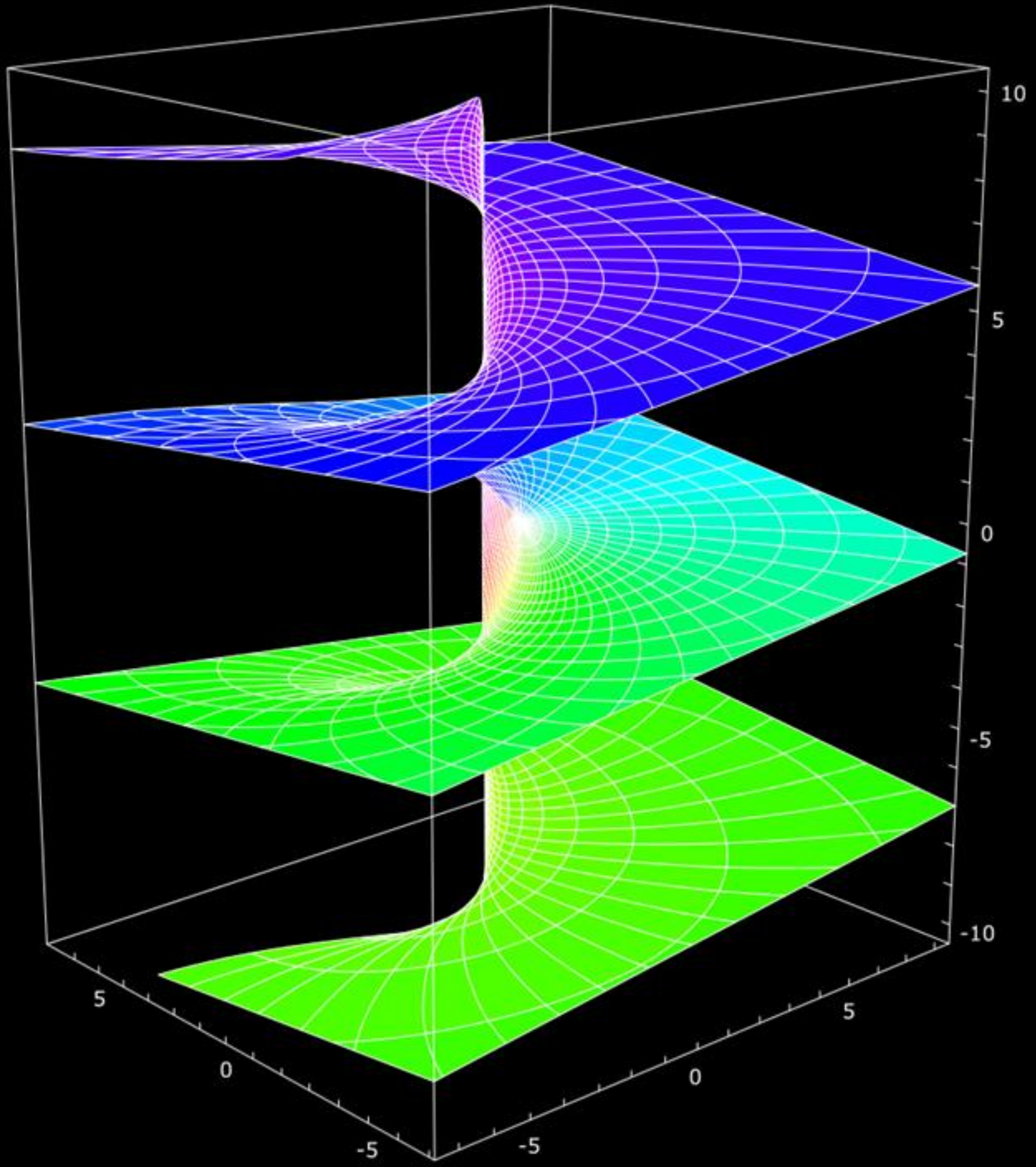
DERIVATIONS

- Tangent vectors ⇒ Left-invariant vector fields ⇒ Directional derivatives
- "Linear operator that satisfies product rule" is equivalent to being a directional derivative for analytic functions



HOW MUCH STRUCTURE IS PRESERVED LOCALLY?

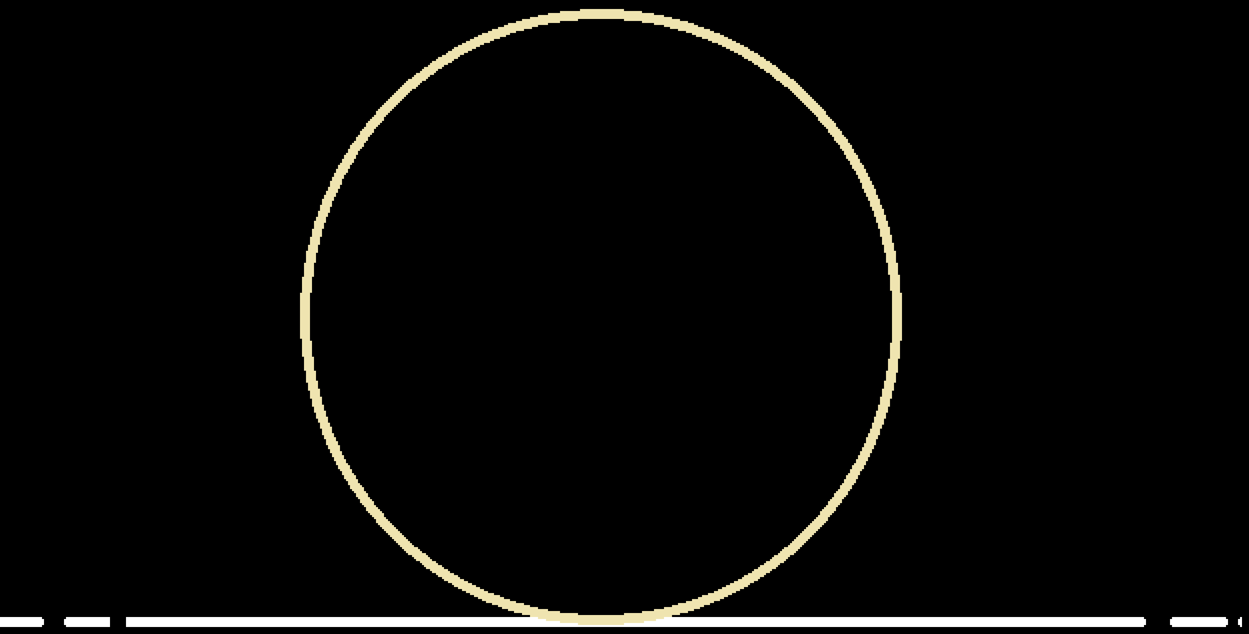
- Sufficiently high-order structure surely determines local structure of group: exp is a local topological isomorphism
- An example of local topological isomorphism is having the same covering space: for connected groups, this is the *only* local Lie group isomorphism



COVERING SPACE (EXAMPLE)

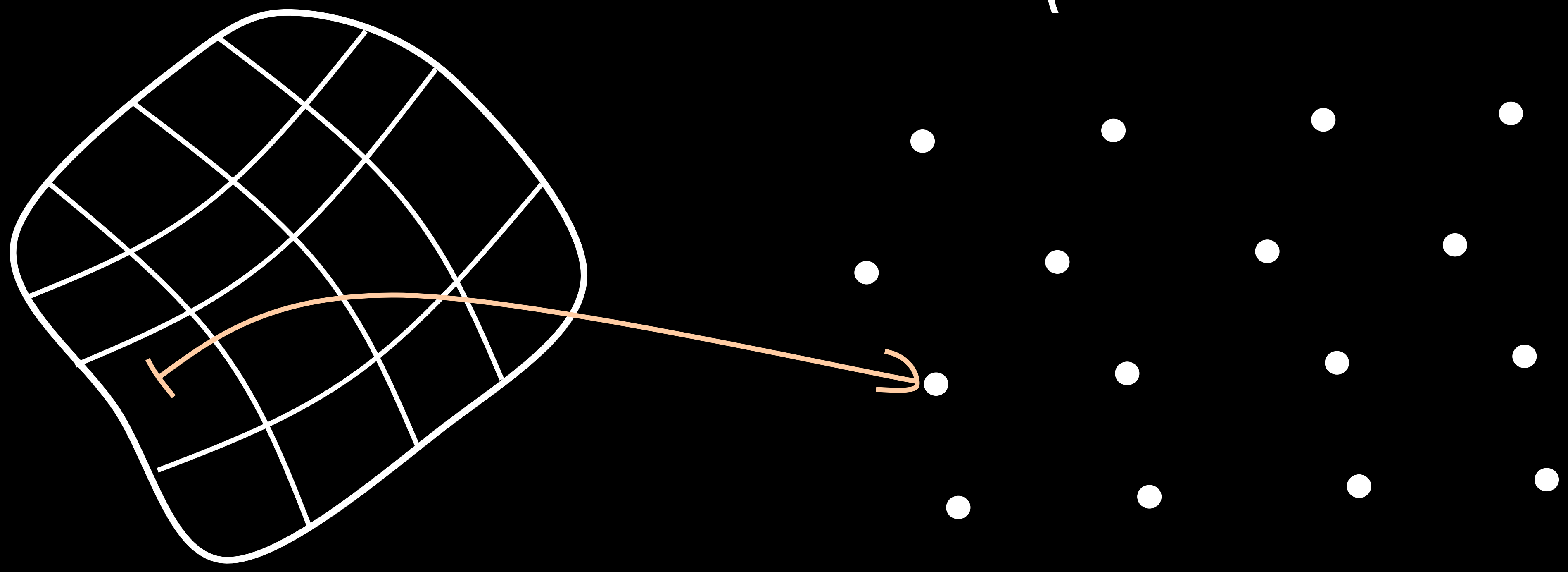
Source:
en.wikipedia.org/wiki/File:
Riemann_surface_log.svg

COVERING SPACE (EXAMPLE)



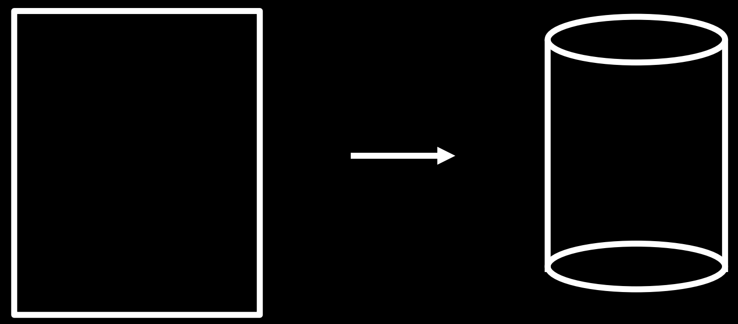
More generally, any plane covers a torus

COVERING SPACE (NON-EXAMPLE)



UNIVERSAL COVER IS SIMPLY-CONNECTED

- Notion of "wrapping": must be a homeomorphism on some open neighbourhood of each point from each connected pre-image
- Any "wrapping" must introduce circles, i.e. eliminate simply-connectedness



- A universal cover is "unwrapping"

DEFINITION OF A LIE ALGEBRA

- Definitions (motivation in Lie's second, third theorems)
  - *Lie algebra* – a vector space equipped with an antisymmetric, bilinear, Jacobi-satisfying product called the "Lie bracket".
  - *Lie algebra homomorphism* – a linear, Lie bracket-preserving map.

- Motivation comes from the *2nd and 3rd fundamental theorems of Lie theory*.

- Definition isn't obvious nonsense: *1st fundamental theorem of Lie theory*.

LIE CORRESPONDENCE

- **Lie's first theorem**
  - Tangent space of a Lie group is a Lie algebra
  - Differential of a Lie group homomorphism is a Lie algebra homomorphism

- **Lie's second theorem –** (G simply-connected) a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ is the differential of a unique Lie group homomorphism $G \to H$

- **Lie's third theorem –** Every Lie algebra is the tangent space of a unique simply-connected Lie group

BAKER-CAMPBELL-HAUSDORFF

$\log(e^X e^Y)$ can be written purely in terms of nested commutators of X and Y

(proof: abhimanyu.io/media/wu-urop.pdf p. 5-6)

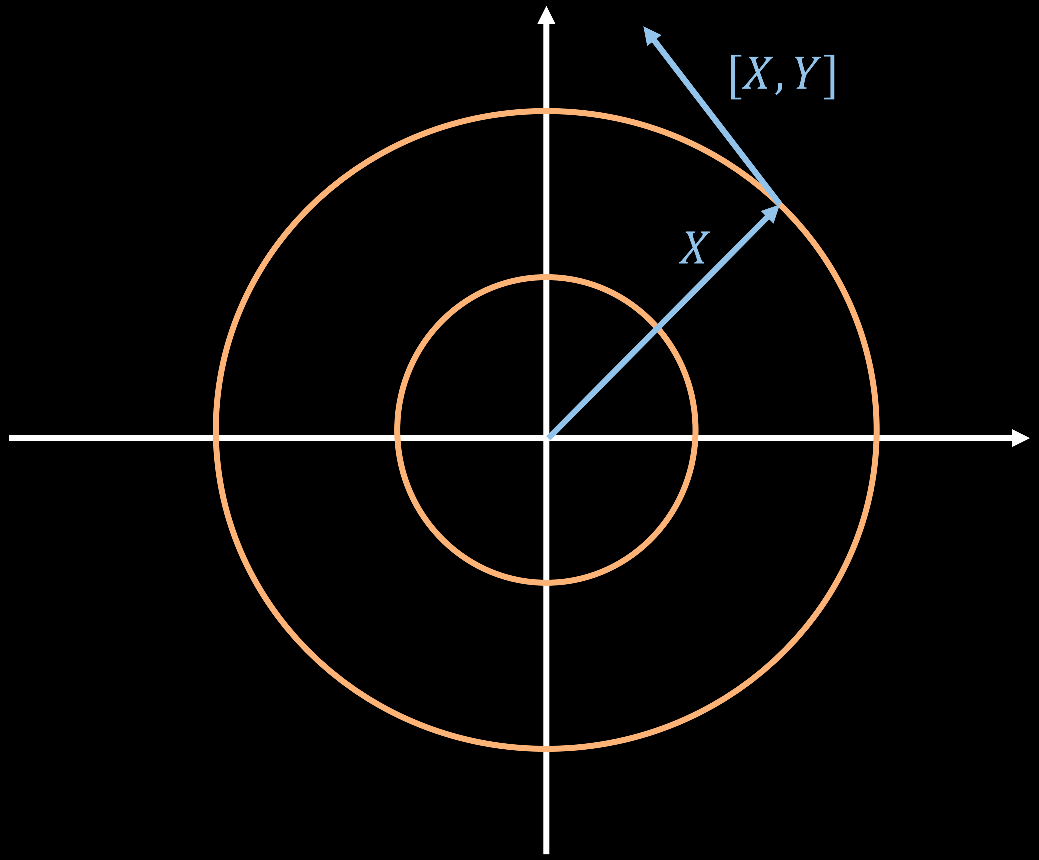
EXAMPLES OF THE CORRESPONDENCE

Lie group : Lie algebra

- Subgroup : Subalgebra
- Normal subgroup : Ideal
- Centre : Centre

KILLING FORM

- Contours of Killing form are orbits of Adjoint map
- Radius perpendicular to tangent: generalisation of $x \cdot (x \times y) = 0$



INDEFINITE ORTHOGONAL GROUP

PARALLEL PARKING

- $\text{Steer} = \frac{\partial}{\partial\theta}$
- $\text{Drive} = \sin(\theta + \phi)\frac{\partial}{\partial x} - \cos(\theta + \phi)\frac{\partial}{\partial y} - \cos\theta\frac{\partial}{\partial\phi}$

- $\left[[\text{Steer}, \text{Drive}], \text{Drive}\right] = \sin\phi\frac{\partial}{\partial x} - \cos\phi\frac{\partial}{\partial y}$