A mathematical definition of property rights

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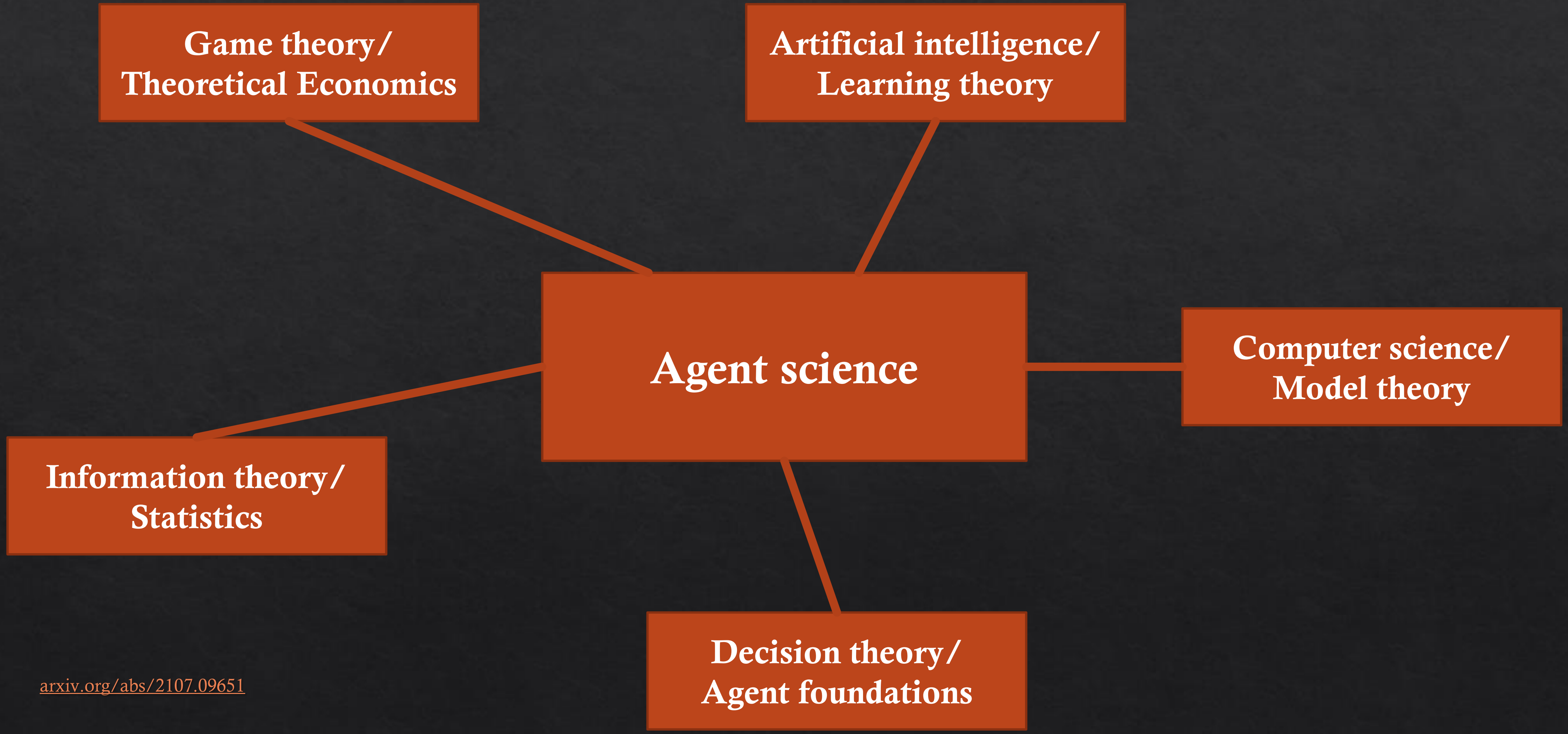
Read through: arxiv.org/abs/2107.09651

"Utility functions are an abstraction of agent behavior."

"Utility functions are coded in, agents attempt to maximize it through efficient heuristics."



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Abstract economy

- Debreu (1952), Debreu & Arrow (1954)
- N agents $\alpha, \beta, \ldots$ and corresponding "choice sets" $\prod_\alpha X_\alpha$, utility functions $U_\alpha : X \to \mathbb{R}$
- Equilibrium exists if:
  - Choice sets are compact, non-empty, convex
  - Supports of utility functions are compact, non-empty, convex
  - Utility functions are continuous, quasi-concave on their support
- Exchange economy arises as a special case, but also other social science models. This is *the* scientific way of doing social science.

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Property rights theory

- Coase (1960), reviewed variously e.g. Randall (1975), Furubotn (1972), Cheung (1970).
- Important insights
  - Multiple "rights structures" are possible that lead to different, but efficient, equilibria
  - Efficiency results from: non-attenuated rights structure, zero transaction costs, perfect information, perfect competition
  - Non-attenuation and zero transaction costs are only defined in a relative sense
- But no formal math

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The mathematics of rights

- **Idea:** exercising a right is a choice to forbid someone else's choice
- We want to say:

$$X_\alpha = X_\alpha^0 \times \prod_\beta 2^{X_\beta}$$

- But this violates Cantor.

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The mathematics of rights

- **Idea:** Think about what a "choice with rights" looks like in English, and formulize that:
  - Me take club from Thrak
  - Me (take club from Thrak AND forbid Thrak take club from me)
  - Me (take club from Thrak AND forbid Thrak (take club from me AND forbid me take club from Thrak))
  - Me (take club from Thrak AND forbid Thrak (take club from Me AND forbid Me (take club from Thrak AND forbid Thrak take club from me))
  - …
- All sentences that make sense – but each successive sentence is more clearly defined than the previous one.
- We intuitively "project" each choice into the previous one – *inverse limits*!

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The mathematics of rights

- Let X^0 be a choice space, and define a sequence (X^n) defined recursively as follows:

$$X_\alpha^{n+1} = X_\alpha^0 \times \prod_\beta 2^{X_\beta^n}$$

- (The β co-ordinate of a choice x_α^n is denoted as $x_{\alpha,\beta}^n$.) And define for $m \leq n$ the projections $\pi_\alpha^{mn}: X_\alpha^n \to X_\alpha^m$ through composition on the following recurrence:

$$\pi_\alpha^{01}\left(x_\alpha^0, R_{\alpha,-\alpha}^0\right) = x_\alpha^0$$
$$\pi_\alpha^{m(m+1)}\left(x_\alpha^0, R_{\alpha,-\alpha}^m\right) = \left(x_\alpha^0, \pi_{-\alpha}^{(m-1)m}\left(R_{\alpha,-\alpha}^m\right)\right)$$

- Then $\pi^{mn}(x) = \left(\pi_\alpha^{mn}(x_\alpha)\right)$ is a family of connecting morphisms under which (X^n) forms an inverse family. The inverse limit $X := \varprojlim X^n$ is called the *consentification* of X^0.

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The mathematics of rights

- Lean code: github.com/abhimanyupallavisudhir/lean/blob/master/rights.lean
- How does this avoid violating Cantor?
- Not any set of choices can be a "forbidden set" – there exist R_β such that $F_\alpha(R_\beta)$ forbids choices not in R_β.
- For $x_\beta = (x_\beta^n) \in X_\beta$, for each m construct an $x_{\beta(m)} \in X_\beta$ such that $x_{\beta(m)}^n = x_\beta^n$ iff $m \le n$. Then x_β is forbidden by $F_\alpha(\{x_{\beta(m)}\})$.

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The mathematics of rights

- For $R \subseteq X_\beta$, define the "closure":

$$\mathrm{cl}(R) = \{y \in X_\beta \mid \forall n, \exists y' \in R, {y'}_n = y_n\}$$

- Kuratowski, T2, first-countable.

- It is precisely the closed sets that may be forbidden!

$$X_\alpha = X_\alpha^0 \times \prod_\beta \phi_\beta$$

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Economics from rights

- Rights structure, non-attenuated rights structure.
- Exchange economy as a consentified economy

$$X_\alpha^0 := \{x_\alpha : A \to \bar{X}_\alpha\}$$

$$U_\alpha^0(x) := \bar{U}_\alpha\left(\sum_{\beta \in A} x_\alpha(\beta)\right)$$

$$U_\alpha(x) = \begin{cases} -\infty & \text{if } \exists y \in x_{\alpha,\beta}, y_0(\alpha) = 0 \\ -\infty & \text{if } \exists \beta, x_\alpha \in x_{\beta,\alpha} \\ -\infty & \text{if } \exists i \leq l, \sum_{\beta \in A} x_{\beta,0}(\alpha)_i > w_{\alpha,i} \\ U_\alpha^0(x_0) & \text{else} \end{cases}$$

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$$R_\beta(p) = \left\{y \in X_\beta \mid \ \left\{x \in X_\alpha \mid p \cdot x_0(\beta) \geq p \cdot y_0(\alpha)\right\} \subseteq y_{\beta,\alpha}\right\}$$

Future work

- Equilibrium and dynamical properties, price theory
- Production economy and transferable rights
- Welfare economics from rights theory

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