Local Normal Forms of Analytic Maps Near Fixed Points

Yuze Jin [*] Jean (Chia-Chun) Lo [†] Anagh Malik [‡] Abhimanyu Pallavi Sudhir [§]
Harmeet Singh [¶]

9th June 2023

Abstract

This report provides an introduction to local behaviour of iterating analytic functions near fixed points and is completed as a part of the M2R group project. We first lay down the foundations for and motivate this study of local behaviour, identifying different classes of dynamics. Throughout each section, we explore these classes one by one and also look to establish various normal forms. Specifically, we will look at conjugating our function to reduce the system to a much easier case from which we can immediately conclude dynamical behaviour. This will involve many notable theorems such as those of Koenigs, Böttcher, Cremer and Siegel, which will allow us to classify a broad range of local behaviour.

[*] yuze.jin18@imperial.ac.uk
[†] chia-chun.lo17@imperial.ac.uk
[‡] anagh.malik18@imperial.ac.uk
[§] abhimanyu.pallavi-sudhir18@imperial.ac.uk
[¶] harmeet.singh18@imperial.ac.uk

Contents

0.1 Symbols Index

Unless stated otherwise, the following notation will be used:

- $\mathbb{C}$, complex plane
- $\widehat{\mathbb{C}}$, Riemann sphere.
- ∂E, boundary of E
- $\overline{E}$, topological closure of E
- f, analytic function of a complex variable.
- $\mathcal{O}(z)$, orbit of z
- z^*, fixed point of f assumed to be 0 without further specification
- $\mathcal{N} \subseteq \mathbb{C}$, neighbourhood of the fixed point z^*.
- $\lambda = f'(z^*)$, first term in the power series expansion of f.
- $f^{\circ n}$, n-fold iterate of f
- $p \in \mathbb{Z}_{\geq 1}$, the smallest integer for which the coefficient of z^{p+1} term in the power series of f is non-zero.
- $\mathbb{D}_r = \{z \in \mathbb{C} \mid |z| < r\}$. We often write $\mathbb{D} = \mathbb{D}_1$
- $\mathbb{R}/\mathbb{Z}$, the circle group consisting of *angles* viewed as the interval $[0, 1)$
- $\mathcal{F}(f)$, the Fatou set of f
- $\mathcal{J}(f)$, the Julia set of f
- $\mathcal{A}_f(z^*)$, basin of attraction of z^* for the function f. If it is obvious what f is, we simply write $\mathcal{A}(z^*)$
- $\mathcal{A}^0(z^*)$, immediate basin of attraction of z^*
- $\mathcal{A}^0(\mathcal{O}, f)$, immediate basin of periodic orbit
- Log, the principal logarithm
- $\mathcal{K}(f)$, the filled Julia set of f

1 Introduction and motivation

Let $f : \mathcal{N} \longrightarrow \mathbb{C}$ be a complex function analytic in some set $\mathcal{N} \subseteq \mathbb{C}$. Suppose further that $z^* \in \mathcal{N}$ is a fixed point of f, $f(z^*) = z^*$ and consider the so-called *forward orbit* of $z \in \mathcal{N}$

$$\mathcal{O}(z) := \left\{z, f(z), f^{\circ 2}(z), \ldots, f^{\circ n}(z), \ldots\right\}$$

(which may be finite or infinite), where

$$f^{\circ n} = \underbrace{f \circ f \circ \ldots f \circ f}_{n}$$

is the n-fold iterate of f so long as this is defined on $\mathcal{N}$. A natural question is what behaviour can we expect for a given $z \in \mathcal{N}$. This question, and more generally the dynamics of this system, have been extensively studied across the twentieth and twenty first century as can be seen in (Cremer 1938), (Siegel 1942), (Milnor 2006) or (Buff and Chéritat 2012). In this work, we will primarily be occupied with understanding local behaviour of such a holomorphic function f around a fixed point z^*. To this end, notions of *normalisation* will be of particular interest, namely in allowing us to classify and determine the dynamics of the above system without having to directly compute iterations of f.

We will often work in $\mathbb{C}$ but at places also consider the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We also assume elementary results from complex analysis and topology. The definition of the Riemann sphere along with certain results can be found in the appendix.

Example 1.1. Suppose $f(z) = \frac{2z^3+1}{3z^2}$. We have fixed points at $z^* = 1, e^{2\pi i/3}, e^{4\pi i/3}$. Then even small perturbations to the initial condition cause changes in behaviour;

$$\lim_{n\to\infty} f^{\circ n}(0.4i) = e^{4\pi i/3}$$
$$\lim_{n\to\infty} f^{\circ n}(0.42i) = e^{2\pi i/3}$$
$$\lim_{n\to\infty} f^{\circ n}(0.415i) = 1$$

This motivates the idea of classifying the dynamical behaviour of an analytic function f so as to determine which regions of the complex plane correspond to what behaviour and construct *normal forms* to classify functions that display certain dynamics. For the above example, see Figure (1).

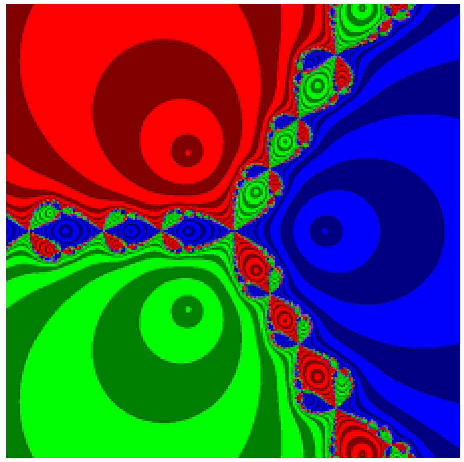

Figure 1: Regions of the complex plane corresponding to initial conditions that converge to one of the three fixed points. Here, $\mathcal{J}(f)$ is the shared boundary consisting of the half-lines $\arg z = \pi, \pi/3, 5\pi/3$ and indeed contains $0.4i$ (Geyer 2016).

We first consider a way of dividing the plane into regions that correspond to *sensitive dependence on the initial condition*. That is, points where small perturbations can result in wildly different behaviour.

Definition 1.2. A family of holomorphic functions F is called *normal* on some domain $\Omega \subseteq \mathbb{C}$ if every sequence of functions in F has a subsequence that is *locally uniformly convergent* on Ω. That is, every sequence in F has a subsequence that converges uniformly on compact subsets $K \subseteq \Omega$.

Definition 1.3. For an analytic function $f : \mathcal{N} \to \mathbb{C}$, the *Fatou set* of f, denoted $\mathcal{F}(f)$, is the largest set on which the family of iterates $\{f^{\circ n}\}$ is normal. The *Julia set*, denoted $\mathcal{J}(f)$, is the complement of the Fatou set, $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$

Here, $z_0 \in \mathcal{J}(f) \iff$ in a neighbourhood of z_0, there is the aforementioned sensitive dependence on the initial point.

Definition 1.4.

1. Let g, h be analytic functions. We say g is *conjugate* to h if there exists a local biholomorphic map ϕ such that
$$\phi \circ g \circ \phi^{-1} = h$$
2. With $f : \mathcal{N} \to \mathbb{C}$ as above, the *multiplier* of a fixed point z^* is defined as $\lambda := f'(z^*)$
3. A point $z \in \mathcal{N}$ is a *periodic point with period* q of f if $f^{\circ q}(z) = z$ and $f^{\circ (q+1)}(z) \neq z$

It is clear that conjugacy is an equivalence relation and the equivalence classes are called *conjugacy classes*. A key property is that the dynamics of our system are preserved under conjugation, more specifically the multiplier λ is independent of the choice of coordinate chart.

Lemma 1.5. *Let $g = \phi \circ f \circ \phi^{-1}$ be conjugate to f. Then*

1. *z^* is a fixed point of $f \iff \phi(z^*)$ is a fixed point of g*

2. *z is a $q-$periodic point of $f \iff \phi(z)$ is a $q-$periodic point of g*

3. *If z^* is a fixed point of f with multiplier λ, then $\phi(z^*)$ is a fixed point of g with multiplier λ*

Proof. 1. $\phi(z^*)$ is a fixed point of $g \iff \phi(f(z^*)) = \phi(z^*) \iff f(z^*) = z^*$. Hence we have a bijection ϕ from the set of fixed points of f to that of g

2. It is clear that $g^{\circ q} = \phi \circ f^{\circ q} \circ \phi^{-1}$. If z is $q-$periodic, then z is a fixed point of $f^{\circ q}$ so $\phi(z)$ is a fixed point of $g^{\circ q}$ by (1). If $g^{\circ (q+1)}(\phi(z)) = \phi(z)$, then $\phi(f^{\circ (q+1)}(z)) = \phi(z) \implies f^{\circ (q+1)}(z) = z$ which contradicts the definition of z being $q-$periodic. The converse holds by a similar argument.

3. This immediately holds by the chain rule since

$$\begin{aligned} g'(\phi(z^*)) &= (\phi \circ f \circ \phi^{-1})'(\phi(z^*))) \\ &= (\phi^{-1})'(\phi(z^*))\phi'(f(z^*))f'(z^*) \\ &= f'(z^*) = \lambda \end{aligned}$$

□

Since the dynamics of f are preserved under conjugation, if z^* is a fixed point of f, we can assume without loss of generality that $z^* = 0$ by conjugating f by a Möbius transformation that carries $z^* \longmapsto 0$. Thus, henceforth unless stated otherwise, we study the behaviour of iterating the analytic function

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 \dots \tag{1}$$

in some neighbourhood $\mathcal{N}$ of the origin.

Specifically, we will come to see that the multiplier $\lambda = f'(0)$ dictates the local behaviour of f above corresponding to the following cases:

1. *Geometrically Attracting or Repelling*: $|\lambda| \notin \{0, 1\}$

2. *Superattracting*: $\lambda = 0$

3. *Parabolic*: $\lambda = e^{2\pi i\xi}$ for rational ξ

4. *Irrationally Indifferent*: $\lambda = e^{2\pi i\xi}$ for irrational ξ

A further classification arises from considering the equivalence relation of conjugacy. As we have seen in a small sense, conjugate maps have qualitatively similar dynamics. With this we hope to find *normal forms* of f, that is simple maps that are conjugate to f, which not only classify the local behaviour of f but also allow us to conclude such dynamics without arduous computation. Finding such normal forms will be a key objective of each following chapter.

2 Geometrically attracting and repelling fixed points

As discussed in the introduction, the value of the *multiplier* λ is of crucial importance in determining the local behaviour of the function. This section will mostly be concerned with the case $|\lambda| \notin \{0, 1\}$. Specifically, we study

1. $0 < |\lambda| < 1$ corresponding to *geometric attraction*, and

2. $|\lambda| > 1$ corresponding to *repelling*

. As before, we study a holomorphic map $f : \mathcal{N} \to \mathbb{C}$ that is analytic in a neighbourhood $\mathcal{N} \subseteq \mathbb{C}$ of the origin which is a fixed point

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots \tag{2}$$

2.1 Attracting Fixed Points

Definition 2.1. The fixed point z^* of f is called *topologically attracting* if $\exists$ a neighbourhood U on which the iterates $f^{\circ n}$ are defined and converge uniformly to the constant map $z \mapsto z^*$.

Remark 2.2. It is important that we demand the convergence to be uniform. Intuitively, this is because we want to imply the notion of a neighbourhood that "shrinks" under iterations of f to an arbitrarily small size.

Lemma 2.3 (Topological Characterization of Attracting Points)**.** *With f as in* (2)*, the origin is topologically attracting* $\iff |\lambda| < 1$.

Proof. As in Eq. 2, we have $f(z) = \lambda z + O\left(z^2\right)$; in particular, there exist constants $r_0 > 0$, $C > 0$ such that for all $|z| < r_0$,

$$|f(z) - \lambda z| \leq C\left|z^2\right|$$

Choose c so that $|\lambda| < c < 1$ and choose $0 < r \leq r_0$ so that $|\lambda| + Cr < c$. Then for $|z| < r$,

$$|f(z)| \leq |\lambda z| + C\left|z^2\right| = (|\lambda| + C|z|)|z| \leq c|z|$$

and so

$$|f^{\circ n}(z)| \leq c^n |z| < c^n r$$

Hence, for any $|z| < r$, $f^{\circ n} \longrightarrow 0$ uniformly as $n \longrightarrow \infty$.

Conversely, if f is topologically attracting, then for any disc $\mathbb{D}_r$ in U there exists some $n > 0$ such that the iterate $f^{\circ n}$ maps $\mathbb{D}_r$ onto a strictly smaller disc $\mathbb{D}_\varepsilon$ as f converges uniformly to a constant map. Applying Cauchy estimates to the derivative, we see that $|(f^{\circ n})'(0)| = |\lambda^n| < 1$, thus $|\lambda| < 1$. □

In the case $|\lambda| \notin \{0, 1\}$, we can establish our first normal form: *local linearisation.* The below Theorem 2.4 will give us a conjugation under which f is locally a linear map, which will lead to incredible simplifications when considering questions of the orbit of the function near the fixed point.

Theorem 2.4 (Koenigs linearisation)**.** *For* $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$ *such that* $|\lambda| \notin \{0, 1\}$, *there exists a local biholomorphic function* $\hat{z} = \phi(z)$ *in some neighbourhood* $\mathcal{N}$ *of 0 satisfying:*

- $\phi(0) = 0$
- $\phi \circ f \circ \phi^{-1}$ *is the linear map* $\hat{z} \mapsto \lambda \hat{z}$ *for* $\hat{z} \in \mathcal{N}$

Moreover, ϕ *is unique up to multiplication by a non-zero constant.*

Proof. We first consider the case where $|\lambda| < 1$.

Let $c < 1$ such that $c^2 < |\lambda| < c$. In the proof of Theorem 2.3, we used a neighbourhood $\mathbb{D}_r$ on which $|f(z)| \leq c\,|z|$. So for some $z_0 \in \mathbb{D}_r$, consider the sequence $z_n = f^{\circ n}(z_0) \longrightarrow 0$ (since $|z_n| \leq rc^n$). Since $|f(z) - \lambda z| \leq C\,|z^2|$ on $\mathbb{D}_r$, we have for z_n that

$$|z_{n+1} - \lambda z_n| \leq C\,|z_n|^2 \leq Cr^2c^{2n}$$

Or writing $k = Cr^2/\,|\lambda|$, and $w_n = z_n/\lambda^n$,

$$|w_{n+1} - w_n| \leq k\left(c^2/\,|\lambda|\right)^n$$

As this converges independently of initial point z_0, the holomorphic function $f^{\circ n}$ converges uniformly on $\mathbb{D}_r$. We define the required function ϕ to be the uniform limit:

$$\phi(z) = \lim_{n \longrightarrow \infty} f^{\circ n}(z)/\lambda^n$$

The required properties are easily satisfied:

- $\phi(0) = 0$ is evident.
- $\phi(f(z)) = \lambda\phi(z)$ follows immediately from $\phi(f(z)) = \lim_{n\longrightarrow\infty} f^{\circ(n+1)}(z)/\lambda^n = \lim_{n\longrightarrow\infty} f^{\circ n}(z)/\lambda^{n-1} = \lambda\phi(z)$

The differentiability of ϕ follows from the Weierstrass Uniform Convergence Theorem, as we have non-zero derivative at the origin for all $f^{\circ n}$, and thus ϕ is locally conformal.

In the case $|\lambda| > 1$, as the fixed point is not critical, we can consider f^{-1} with multiplier $\left|\lambda^{-1}\right| < 1$, and reduce the problem to a solved one.

To prove uniqueness, consider some alternative Koenigs linearisation $\tilde{\phi}$. Then for the map $\tilde{\phi} \circ \phi^{-1}(\hat{z})$:

$$\lambda\tilde{\phi} \circ \phi^{-1}(\hat{z}) = \tilde{\phi} \circ f \circ \phi^{-1}(\hat{z}) = \tilde{\phi} \circ \phi^{-1}(\lambda\hat{z})$$

Expanding

$$\tilde{\phi} \circ \phi^{-1}(\hat{z}) = b_1\hat{z} + b_2\hat{z}^2 + b_3\hat{z}^3 + \dots$$

Compare the coefficient of $\hat{z}^n$ on both sides we can get $\lambda b_n = b_n\lambda^n$, As $|\lambda|$ is not 0 or 1, $b_n = 0$ for all $n \neq 1$, implying that $\phi \circ \phi^{-1}$ is just multiplication by a constant. □

Remark 2.5. We can generalise the statement above by changing the coefficients b_n to $b_n(\alpha)$. the map can be written as

$$f_\alpha(z) = \lambda(\alpha)z + b_2(\alpha)z^2 + \dots$$

Which depends on a parameters $\alpha \in \mathbb{C}$ with the required $|\lambda(\alpha)| \neq 0, 1$ as above. Then the Koenigs linearisation $\phi(z) = \phi_\alpha(z)$ is still valid and is dependent on α. If we first fix some $0 < c < 1$ and suppose that $|\lambda(\alpha)|$ only takes values in some compact subset of the interval $\left(c^2, c\right)$, then as per the same proof as above, we can show that $f^{\circ n}$ converges uniformly. With c arbitrary, we have the general Koenigs linearisation.

In the case of a geometrically attracting fixed point, we can extend our conjugating map ϕ from Theorem (2.4) to the so called basin of attraction $\mathcal{A} = \mathcal{A}(0)$.

Definition 2.6. The *attraction basin* $\mathcal{A}(z^*)$ of a fixed point z^* is the set of all points that converge to z^* under iterations of f

$$\mathcal{A}(z^*) = \{z_0 \mid \lim_{n\to\infty} f^{\circ n}(z_0) = z^*\}$$

The *immediate basin* $\mathcal{A}^0(z^*)$ is the connected component of $\mathcal{A}(z^*)$ that contains z^*.

Theorem 2.7 (Global linearisation)**.** *Up to multiplication by a non-zero constant, there exists a unique local biholomorphic map* $\phi : \mathcal{A} \to \mathbb{C}$ *satisfying the criteria of Theorem* (2.4)*.*

Proof. Without loss of generality we can take the fixed point to be the origin. From Theorem 2.4, we know $\exists\ r > 0$ with a unique local Koenigs linearisation $\phi_r(z)$ on $\mathbb{D}_r$. In particular, this linearisation satisfies $\phi_r(z) = \lambda^{-n}\phi_r(f^{\circ n}(z))$ by a simple induction.

We can use this property to extend ϕ_r to the basin of attraction. For any $z \in \mathcal{A}$, $f^{\circ n}(z) \in \mathbb{D}_r$ for all but finitely many n. Thus we can define $\phi(z) = \lambda^{-n}\phi_r(f^{\circ n}(z))$ where n is the smallest integer for which $|f^{\circ n}(z)| < r$. Taking $n = 0$ we have the extended function coincide the local one on $\mathbb{D}_r$.

It's trivial to see that our required properties are satisfied. Indeed

$$\phi(f(z)) = \lambda^{-n}\phi_r(f^{\circ(n+1)}(z)) = \lambda^{-n}\lambda^{n+1}\phi(z) = \lambda\phi(z)$$

as we require, so the global linearisation is indeed an analytic continuation of the local one.

Uniqueness follows directly from local uniqueness, since any global linearisation $\tilde{\phi}$ must satisfy $\tilde{\phi}(f^{\circ n}(z)) = \lambda^n\tilde{\phi}(z)$, i.e. it must be determined by the local unique linearisation on $\mathbb{D}_r$. □

The global linearisation allows us to find a *curve of attraction*; that is a curve in $\mathcal{A}$ that is invariant under f and contains the fixed point. Such a curve clearly for the linear map $z \mapsto \lambda z$, and is a logarithmic spiral. The preimage of this curve under ϕ then gives a curve of attraction for f.

Consider now the special case when f is a rational function over $\widehat{\mathbb{C}}$. That is, f is of the form

$$f(z) = \frac{P(z)}{Q(z)}$$

where P and Q are polynomials in z with no common factor. We say that f has *degree* $d \in \mathbb{N}$ where $d = \max\{\deg(P), \deg(Q)\}$.

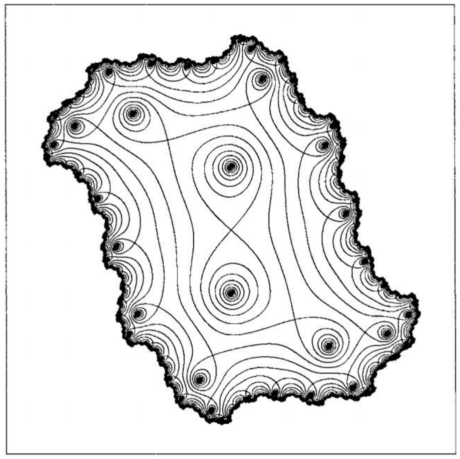

Figure 2: [(Milnor 2006), p.80]Julia set for $f(z) = z^2 + 0.7iz$. The critical point $-0.35i$ is at the centre of the figure, and the origin is the centre of the nested circles directly above it. We see that the upper half of the figure eight bounds the region $\psi(\mathbb{D}_r)$ described below.

To simplify the problem we only consider the case that the fixed point $z^* \in \mathbb{C}$ here so that we can directly find a local linearising function ϕ and f is a function with $d \geq 2$.

The same as (2.7), in some neighbourhood $\mathbb{D}_\varepsilon$, we still have a local inverse $\psi_\varepsilon : \mathbb{D}_\varepsilon(0) \to \mathcal{A}^0$ to the linearising map $\phi : \mathcal{A} \to \mathbb{C}$.

Lemma 2.8. *This local inverse $\psi_\varepsilon : \mathbb{D}_\varepsilon \to \mathcal{A}^0$ can be uniquely analytically extended to some maximal open disc $\mathbb{D}_r$ as $\psi_r : \mathbb{D}_r \to \mathcal{A}^0$ with $\psi_r(0) = 0$ and $\phi \circ \psi_r(\hat{z}) = \hat{z}$.*

Furthermore, ψ_r can be continuously extended to the boundary $\partial\mathbb{D}_r$ and there exists at least one critical point of f in the $\psi_r(\partial\mathbb{D}_r)$.

Proof. First observe that ψ_ε is an inverse of ϕ in some neighbourhood of origin. As the derivative of ϕ at zero is 1 such local ψ_ε exists. In addition, we have by the permanence principle that any holomorphic extension of ψ_ε is still an inverse of ϕ on its domain.

We now show that a finite "maximal radius" exists, i.e. an extension ψ of ψ_ε cannot be defined on the entire complex plane: recall from the proof of Lemma 2.7 that $\phi(z) = \lambda^{-n}\phi(f^{\circ n}(z))$, so $f^{\circ n}(\psi(\hat{z})) = \psi(\lambda^n \hat{z})$. Letting $n \longrightarrow \infty$ we have $\psi(0) = 0$, which tells us that any extension ψ of ψ_ε has its codomain in the basin of attraction $\mathcal{A}$. In particular, it maps any disc $\mathbb{D}_R$ into $\mathcal{A}^0$, due to the connectedness of $\mathbb{D}_R$. Now suppose that ψ is defined on the entire complex plane: $\psi : \mathbb{C} \to \mathcal{A}^0$. Now, $\mathcal{A}^0$ omits more than three points from $\widehat{\mathbb{C}}$ hence is hyperbolic(It omits the whole Julia set which is non empty and no isolated point (Milnor 2006) and thus by Picard's theorem ψ would be a constant map, which is impossible as 0 is not a critical point.

Thus there exists a maximal open $\mathbb{D}_r$ on which the extension $\psi = \psi_r$ is defined. Where U is the open set $\psi(\mathbb{D}_r) \subset \mathcal{A}^0$, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \\ \psi \big\uparrow \big\downarrow \phi & & \psi \big\uparrow \big\downarrow \phi \\ \mathbb{D}_r & \xrightarrow[\lambda\cdot]{} & \lambda\mathbb{D}_r \end{array}$$

Next, we show that the image of the boundary of the maximal domain $\mathbb{D}_r$ must contain a critical point. We prove this by contradiction, assume there is no critical point on the image of the boundary of the maximal domain $\mathbb{D}_r$. We have $f(\overline{U}) \subset \overline{f(U)} = \overline{\psi(\mathbb{D}_R)} \subset U$, implying that $\overline{U}$ is mapped to U under f. Since $U \subseteq \mathcal{A}^0$, this means $\overline{U} \subseteq \mathcal{A}^0$. So we have ϕ defined in some neighbourhood of $\overline{U}$. Differentiating $\phi(f(z)) = \lambda\phi(z)$, we have $\tilde{\phi}(f(z))f'(z) = \lambda\tilde{\phi}(z)$. Now for any z which is *not* a critical point, $f'(z) \neq 0$. As ϕ is injective in U $\tilde{\phi}(f(z)) \neq 0$, $\tilde{\phi}(z) \neq 0$, thus ϕ has a local inverse on some open disc about $\hat{z} = \phi(z)$. The union of these discs covers the boundary of $\mathbb{D}_r$ and so by compactness of $\partial\mathbb{D}_r$, a finite union covers $\partial\mathbb{D}_r$, and thus the domain of the inverse can be extended to a strictly larger disc, contradicting maximality of $\mathbb{D}_r$.

Thus there must exist at least one critical point on the boundary of image of $\mathbb{D}_r$.

Finally, we construct an extension of ψ_r to the boundary. Choosing a sequence $(\hat{z}_n)$ on $\mathbb{D}_r$ converging to some point $\hat{z}_\infty \in \partial\mathbb{D}_R$, we have $f(\psi_r(\hat{z}_n)) = \psi_r(\lambda\hat{z}_n) \longrightarrow \psi_r(\lambda\hat{z}_\infty)$ as $n \longrightarrow \infty$. Since $|\lambda| < 1$, $\lambda\hat{z}_\infty \in \mathbb{D}_r$, $z = \psi_r(\lambda\hat{z}_\infty)$ is well-defined. Thus every convergent subsequence of $\psi_r(\hat{z}_n)$ converges to some point in the finite set $f^{-1}(\{z\})$, and the set of limit points of $\psi(\hat{z})$ as $\hat{z} \longrightarrow \hat{z}_\infty$ is a subset of $f^{-1}(\{z\})$ by the argument above. Since the set of limit points of $\psi(\hat{z})$ as $\hat{z} \longrightarrow \hat{z}_\infty$ is connected(Consider any two points in this set $\psi(\hat{z}_{1_\infty})$ and $\psi(\hat{z}_{2_\infty})$ where $\hat{z}_{1_n} \to \hat{z}_{1_\infty}$, $\hat{z}_{2_n} \to \hat{z}_{2_\infty}$. Then consider the image of the line segment S_n joining $\hat{z}_{1_n}$ and $\hat{z}_{2_n}$ under ψ and this is connected So if we let $n \to \infty$ we see $\psi(S_\infty)$ is connected and contain $\psi(\hat{z}_{1_\infty})$ and $\psi(\hat{z}_{2_\infty})$. As this is true for any two points in this set the whole set is connected) , it contains exactly one element, namely z_0. Thus any convergent sequence $\psi_r(\hat{z}_n)$ converges to the same limit for any sequence $\hat{z}_n \to \hat{z}_\infty$. Thus this extension $\psi_r(\hat{z}_\infty) = z_0$ is continuous. □

2.2 Attracting Periodic Orbits

Definition 2.9. A *periodic orbit* is an orbit $z_0 \to z_1 \to z_2 \to \cdots$ such that $z_m = f^{\circ m}(z_0) = z_0$ for some integer m. A periodic orbit is called *attracting* if the derivative $\left|(f^{\circ m})'(z_k)\right| < 1$.

(Note that in the complex plane, this derivative is equal, by the chain rule, to $\prod f'(z_k)$ thus is same for all z_i. This independence of z_i is also true on Riemann sphere.)

A periodic orbit can be viewed as a generalisation of the notion of a fixed point: a periodic point of f of period m is a fixed point of the iterate $f^{\circ m}$. Accordingly we define the immediate basin of a periodic orbit; intuitively, this captures the notion of points which asymptotically 'wrap around' the periodic orbit:

Definition 2.10. Since each z_k is a fixed point of $f^{\circ m}$, they have corresponding immediate basins. The immediate basin $\mathcal{A}^0(f, \mathcal{O})$ of a periodic orbit $\mathcal{O}$ is the union of the immediate basins of each point in the orbit under $f^{\circ m}$.

Theorem 2.11. *For f a nonlinear rational map, the immediate basin of every attracting periodic orbit contains at least one critical point.*

Proof. Clearly, $f\left(\mathcal{A}^0\left(z_j\right)\right) \subset \mathcal{A}^0\left(z_{j+1}\right)$ (any point that goes arbitrarily close to z_j will be close to z_{j+1} after one iterate). Inductively applying f, we have that $f^{\circ m}$ will map each $\mathcal{A}^0\left(z_j\right)$ into itself.

Suppose there is no critical point in $\mathcal{A}^0(f, \mathcal{O})$, i.e. no critical point in any of $\mathcal{A}^0\left(z_j\right)$. By chain rule, this means that there is no critical point of $f^{\circ m}$ in any of $\mathcal{A}^0\left(z_j\right)$.

However, this is easily seen to be false: consider some point $z_j \in \mathcal{O}$. As the orbit is attracting, this point is attracting with respect to $f^{\circ m}$. If it is super-attracting, it is itself a critical point, and if it is geometrically attracting, there is a critical point in $\mathcal{A}^0(z_j)$ by Lemma 2.8. Thus we must have a critical point in $\mathcal{A}^0(f, \mathcal{O})$. □

Corollary 2.12. *Such a rational map f has only finitely many attracting periodic orbits.*

Proof. Since the immediate basin of different attracting periodic orbits are clearly disjoint and f can have only finitely many critical points, the result follows. □

Remark 2.13. Theorem 2.11 is useful in approximating the periodic orbits of a rational map, as one may first locate all critical points of the function and iteratively apply the function from the critical point and observe if it converges to a periodic orbit. However, this algorithm may be invalid for orbits with large periods (such as the map $f(z) = z^2 - 1.5$ for example).

2.3 Repelling Fixed Points

We now consider repelling points: those points with multiplier $|\lambda| > 1$.

Definition 2.14. The fixed point z^* of f is called *topologically repelling* if for some neighbourhood $\mathcal{N}$ of z^*, $\forall z \in \mathcal{N}$ and $z \neq z^*$, $\exists n \in \mathbb{N}$ s.t. $f^{\circ n}(z)$ leaves $\mathcal{N}$. Thus the only orbit that stays in $\mathcal{N}$ is the orbit of the fixed point z^*. Here we call $\mathcal{N}$ a *forward isolating neighbourhood* of z^*.

For holomorphic maps on the complex plane, we have a similar result to Lemma 2.3.

Lemma 2.15. *The fixed point* 0 *of* $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$ *is topologically repelling if and only if its multiplier satisfies* $|\lambda| > 1$.

Proof. As in Eq. 2, we have $f(z) = \lambda z + O(z^2)$ so there exist constants $r > 0$, $C > 0$ such that $\forall\, |z| < r$,

$$|f(z) - \lambda z| \leq C \left|z^2\right|.$$

If we take $0 < r_0 \leq r$ sufficiently small so that $c := |\lambda| - Cr_0 > 1$ and set $\mathcal{N} = \mathbb{D}_{r_0}$, then $\forall\, z \in \mathcal{N}$,

$$|f(z)| \geq |\lambda z| - C\left|z^2\right| = c\,|z|$$

$$\Longrightarrow |f^{\circ n}(z)| \geq c^n\,|z|.$$

As, the RHS of the final inequality $\longrightarrow \infty$ as $n \longrightarrow \infty, \forall z \in \mathcal{N} \setminus \{0\}$ since $c > 1$. Thus, $\exists\, n > 0$ s.t. $f^{\circ n}(z)$ will $\mathcal{N}$ for $z \neq 0$, as required.

Conversely, suppose $\hat{p}$ is topologically repelling fixed point of f. Then certainly $\hat{p}$ is *not* topologically attracting and so by Lemma 2.3, $\lambda = f'(\hat{p}) \neq 0$. Thus, by the Inverse Function Theorem, we can choose some compact forward isolating neighbourhood $\mathcal{N}$ of $\hat{p}$ which f maps homeomorphically onto a compact neighbourhood $f(\mathcal{N})$. Define

$$\mathcal{N}^k = N \cap f^{-1}(\mathcal{N}) \cap (f^{-1})^{\circ 2}(\mathcal{N}) \cap \cdots \cap (f^{-1})^{\circ k}(\mathcal{N})$$

So we have $\mathcal{N} \supset \mathcal{N}^1 \supset \mathcal{N}^2 \supset \dots$ is a nested sequence of compact sets all containing $\hat{p}$. Since $\hat{p}$ is topologically repelling, the intersection of these nested sets contains only the fixed point thus its diameter tends to 0.

From our construction, we immediately have $f(\mathcal{N}^k) = \mathcal{N}^{k-1} \cap f(\mathcal{N})$ and observing that $\mathcal{N}^k \subseteq f(\mathcal{N})$ for sufficiently large k (because the diameter of $\mathcal{N}^k$ tends to 0), we have $f(\mathcal{N}^k) = \mathcal{N}^{k-1}$ for suffiently large k. If $\mathcal{N}_0^k$ denotes the connected component of $\hat{p}$ in $\mathcal{N}^k$, we see that f^{-1} maps $\mathcal{N}_0^{k-1}$ biholomorphically onto the strictly smaller set $\mathcal{N}_0^k$. By Schwarz's lemma, the multiplier of f^{-1} is $< 1 \Longrightarrow \left|\lambda^{-1}\right| < 1 \Longrightarrow |\lambda| > 1$. ∎

One may imagine that the global linearisation in the repelling case is similar to that in the attracting case. However, there isn't really such a thing as a "repelling basin" – instead we can generalise ψ_ε to the entire complex plane. Here we consider $f : \mathbb{C} \to \mathbb{C}$.

Theorem 2.16. *For a repelling fixed point of f, there exists an entire bijective function ψ such that $\psi(0) = 0$ and ψ conjugates f to the linear map $\hat{z} \mapsto \lambda\hat{z}$. Moreover, ψ is unique (up to multiplication by a non-zero constant).*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C} \\ {\scriptstyle\psi}\uparrow & & {\scriptstyle\psi}\uparrow \\ \mathbb{C} & \xrightarrow[\lambda]{} & \mathbb{C} \end{array}$$

Proof. Recall from theorem 2.4, we have existence of ϕ and as 0 is not a critical point we have from the inverse function theorem that on some neighbourhood $\mathbb{D}_\varepsilon(0)$, there exists a unique inverse ψ_ε with $\psi_\varepsilon(0) = 0$.

Let $z \in \mathbb{C}$, and choose the smallest n such that $z/\lambda^n \in \mathbb{D}_\varepsilon(0)$, then define $\psi(z) = f^{\circ n}(\psi_\varepsilon(z/\lambda^n))$ and the required properties and uniqueness can be verified as in Lemma 2.8. □

3 Superattracting fixed points

3.1 Introduction

In this section we will be dealing with *superattracting* fixed points i.e. where the multiplier $\lambda = 0$. As before we can assume without loss of generality our map has a fixed point at 0 and so has the form

$$f(z) = a_p z^p + a_{p+1} z^{p+1} \cdots = \sum_{k=p}^{\infty} a_k z^k \tag{3}$$

where $p \geq 2$, $a_p \neq 0$. Here p is called the *local degree*.

3.2 Böttcher's Theorem

It is reasonable to expect that we cannot define a *linearisation* near superattracting fixed points, as the linear aspect of our function is zero. We thus look to *higher derivatives* of f at the fixed point in order to characterise the local behaviour. More generally, we consider a substitution that causes f to locally behave like the power map $z \mapsto z^p$ near the fixed point.

Theorem 3.1. *Let f be as in Eq. 3. Then there exists a holomorphic change of coordinates $\hat{z} = \phi(z)$, such that $\phi(0) = 0$ and $\phi'(0) = 1$, where $\phi(f(z)) = \phi(z)^p$ in a neighbourhood of 0. Furthermore, ϕ is unique up to multiplication by a $(p-1)^{th}$ root of unity.*

Proof. For convenience, let $h(z) = cz$ where c is some $(p-1)^{th}$ root of a_p. Then h conjugates f to a simpler form

$$\left(h \circ f \circ h^{-1}\right)(z) = cf(z/c) = c \sum_{k=p}^{\infty} a_k z^k / c^k = z^p + \sum_{k=p+1}^{\infty} a_k z^k / c^k = z^p + \sum_{k=p+1}^{\infty} b_k z^k$$

Hence, we may assume without loss of generality that

$$f(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k = z^p (1 + \eta(z))$$

where $\eta(z) = \sum_{k=1}^{\infty} b_k z^k$.

Now let $0 < r < \frac{1}{2}$ be sufficiently small so that $\forall z \in \mathbb{D}_r$, $|\eta(z)| < \frac{1}{2}$. Then if $z \in \mathbb{D}_r$, we have

$$|f(z)| = |z^p(1+\eta(z))| < \frac{3}{2}|z|^p \leq \frac{3}{4}|z|$$

since $|z| < \frac{1}{2}$. Hence f maps $\mathbb{D}_r$ into itself, with $f(z) \neq 0\ \forall z \in \mathbb{D}_r \setminus \{0\}$. It follows then that $\mathbb{D}_r$ is invariant under any iterate $f^{\circ k}$.

Claim. $f^{\circ k}(z) = z^{p^k}(1 + p^{k-1}b_1 z + z^2 q(z))$ for some polynomial q in z.

Proof. This is clearly the case for $k = 1$, and assuming the statement for k, we see:

$$\begin{aligned}
f^{\circ(k+1)}(z) &= f(f^{\circ k}(z)) \\
&= f(z^{p^k}(1 + p^{k-1}b_1 z + z^2 q(z))) \\
&= z^{p^{k+1}}(1 + p^{k-1}b_1 z + z^2 q(z))^p(1 + \eta(f^{\circ k})) \\
&= z^{p^{k+1}}(1 + p^k b_1 z + z^2 r(x))
\end{aligned}$$

□

We now define $\phi_k : \mathbb{D}_r \to \mathbb{C}$ for $k = 1, 2 \ldots$ which will converge uniformly to our ϕ on $\mathbb{D}_r$. Specifically,

$$\phi_k(z) = (f^{\circ k}(z))^{\frac{1}{p^k}} = z(1 + p^{k-1}b_1 z + z^2 q(z))^{\frac{1}{p^k}}$$

where we take the unique $p^k - th$ root so that the Taylor expansion of $\phi_k(z)$ about 0 is of the form

$$\phi_k(z) = z(1 + \frac{b_1}{p}z + \ldots)$$

This gives us the recursive relation $\phi_k(f(z)) = (f^{\circ(k+1)}(z))^{\frac{1}{p^k}} = \phi_{k+1}(z)^p$. We make the substitution $\tilde{z} = \omega(z) := \log(z)$. Due to the restriction of z to $\mathbb{D}_r$, we have $Re(\tilde{z}) < \log(r)$. Let $\mathbb{H}_r$ denote the set of points that satisfy this inequality. Then ω conjugates f to $\tilde{f}$, where

$$\tilde{f}(\tilde{z}) = \log(f(e^{\tilde{z}}))$$
$$= \log(e^{p\tilde{z}}(1 + \eta(e^{\tilde{z}}))$$
$$= p\tilde{z} + \mathrm{Log}(1 + \eta(e^{\tilde{z}}))$$

Note we can take the principal branch of the logarithm, since $1 + \eta$ is contained in the ball centred at 1 of radius $\frac{1}{2}$ and hence $\tilde{f}$ is a holomorphic function $\mathbb{H}_r \to \mathbb{H}_r$ (due to the invariance of $\mathbb{D}_r$ under f). Now we also have:

$$\left|\tilde{f}(\tilde{z}) - p\tilde{z}\right| = |\log(1 + \eta)| \leq \log(2) < 1 \tag{4}$$

since $|\eta| < \frac{1}{2}$, as established previously. Hence, ω also conjugates ϕ_k to $\tilde{\phi}_k$, where $\tilde{\phi}_k(\tilde{z}) = \log(\phi_k(e^{\tilde{z}})) = \frac{\tilde{f}^k(\tilde{z})}{p^k}$. This is also a holomorphic function $\mathbb{H}_r \to \mathbb{H}_r$. Now using Eq. 4, we have:

$$\left|\tilde{\phi}_{k+1}(\tilde{z}) - \tilde{\phi}_k(\tilde{z})\right| = \left|\frac{\tilde{f}^{\circ(k+1)} - p\tilde{f}^{\circ k}(\tilde{z})}{p^{k+1}}\right| < \frac{1}{p^{k+1}}$$

Then if $k > l$:

$$\left|\tilde{\phi}_k(\hat{z}) - \tilde{\phi}_l(\hat{z})\right| \leq \frac{1}{p^k} + \frac{1}{p^{k-1}} + \cdots + \frac{1}{p^{l+1}} < \frac{1}{p^{l+1}}\frac{p}{p-1}$$

We see that the final inequality $\longrightarrow 0$ as $l \longrightarrow \infty$. Hence, it follows that $\{\tilde{\phi}_k\}_{k\in\mathbb{N}}$ is a uniformly convergent sequence because of the Cauchy property.

Now the exponential function is a contraction on $\mathbb{H}_r$, since $|e^z| = \left|e^{a+bi}\right| = |e^a|$, and $\mathbb{H}_r$ is a subset of the left half plane (the result of contraction follows simply from ML inequality). Hence we have that ϕ_k is also a Cauchy sequence i.e. uniformly convergent.

Now with the uniform convergence of ϕ_k, where $k = 1, 2\ldots$ we can take the limit from both sides of the recursive relation to end up $\phi(f(z)) = \phi(z)^p$. It is also easy to see that $\phi(0) = 0$ and $\phi'(0) = 1$ (this will help use define an inverse later on). Hence we are done with existence.

Now to prove **uniqueness**. Firstly using the Inverse Function Theorem we notice that since $\phi'(0) = 1$, there is an open subsets U containg 0, so that ϕ is invertible on the restriction to U. Now if two maps ϕ_1 and ϕ_2 conjugate f to the power map in some neighbourhood of 0, then $\phi_1 \circ \phi_2^{-1}$ conjugates the power map to itself in some neighbourhood of 0. So now we can limit ourselves to the case when $f(z) = z^p$, suppose $\phi(z) = c_1 z + c_k z^k \dots$ conjugates f to itself, then we have $\phi(f(z)) = \phi(z)^p$ i.e:

$$c_1 z^p + c_k z^{pk} \dots = c_1^p z^p + p c_1^{p-1} c_k z^{p+k-1} \dots$$

comparing coefficients, we see $c_1^{p-1} = 1$ and all higher coefficients are 0 (since degrees don't match up). And so we see $\phi_1 = c_1 \phi_2$ and hence we are done.

□

Notice we can't extend ϕ throughout the basin of attraction if we want a biholomorphic map, if for example a point gets mapped into the fixed point by f, we will run into problems with the way we defined ϕ. We can however extend the absolute value of ϕ:

Theorem 3.2. *Let f, ϕ be as in Theorem 3.1 with basin of attraction of 0, $\mathcal{A}$. Then the function $|\phi|$ extends uniquely to a continuous map $|\phi| : A \to [0, 1)$, which satisfies the identity $|\phi(f(z))| = |\phi(z)|^n$.*

Proof. Let ϕ_0 be the map from Theorem 3.1, defined on $K \subset \mathcal{A}$. For every $z \in \mathcal{A}$, there exists k s.t $f^{\circ k}(z) \in K$, by definition of basin of attraction. Then, for any $z \in \mathcal{A}$ and such a k, we define:

$$|\phi(z)| := \left|\phi_0(f^{\circ k}(z))\right|^{\frac{1}{n^k}}$$

This is the unique continuous extension of $|\phi_0|$.

To show $|\phi|$ is continuous fix any $r \in \mathcal{A}$. Let k be such that $f^{\circ k}(r) \in K$. Then due to continuity of $f^{\circ k}$, there is a neighbourhood of r, which also gets mapped into K then the result follows by the composition of continuous functions. Also notice, since $\phi(0) = 0$ we have $|\phi(z)|^{p^k} = |\phi(f^{\circ p}(z))| \longrightarrow 0$ (where p is the local degree of f), so we must have $0 \leq |\phi(z)| < 1$. □

Now we can ruminate on how far ϕ can be extended. Of course this is not meaningful to generalize since as mentioned before, we could have a point mapped to the fixed point by f in the basin. It is hence more meaningful to discuss the extension of the local inverse of ϕ, which we have already understood exists, this follows from the Inverse Function Theorem and the fact that $\phi'(0) = 1$. If ψ_r is the local inverse, we will extended it to ψ with the theorem below:

Theorem 3.3. *Let f and ϕ be as in Theorem 3.1 and let ψ_r be inverse of ϕ defined on an open neighbourhood of 0, V . Then there exists a unique open disc $\mathbb{D}_\varepsilon$ around 0 of maximal radius $0 < \varepsilon \leq 1$ such that ψ_r extends holomorphically to a map ψ from the disc into the immediate basin $\mathcal{A}^0$ of 0. If $\varepsilon = 1$, then ψ maps the unit disc biholomorphically onto $\mathcal{A}^0$ and 0 is the only critical point of f in the basin. On the other hand if $\varepsilon < 1$ then there is at least one other critical point of f in $\mathcal{A}^0$, lying on the boundary of $\psi(\mathbb{D}_\varepsilon)$.*

Proof. By holomorphic continuation we can extend ψ_r as stated in the theorem to some disc of radius $\varepsilon \leq 1$. Then we have by Theorem 3.1, $q(z) := \psi(z^p) - f(\psi(z)) = 0$ on V and since q is holomorphic, we can use the Permanence Principle 6.3 from which we deduce $q = 0$ on $\mathbb{D}_\varepsilon$, similarly we get for all k, $f^{\circ k}(\psi(\hat{z})) = \psi\left(\hat{z}^{p^k}\right)$. From this we get (since $|\hat{z}| < 1$), as as $k \longrightarrow \infty$:

$$f^{\circ k}(\psi(\hat{z})) = \psi(\hat{z}^{p^k}) \longrightarrow \psi(0) = 0$$

Combining that with the fact that ψ is continuous and $\mathbb{D}_\varepsilon$ is connected, we get $\psi(\mathbb{D}_\varepsilon) = U \subset \mathcal{A}^0$, since $0 \in U$.

We will now show that ψ is conformal and injective, then for the case $\varepsilon = 1$ we will show it maps onto the whole immediate basin, this with the Inverse Function Theorem gives biholomorphism for that case.

Firstly suppose that $\psi'(\hat{z}) = 0$, for some $\hat{z}$. Then by chain rule and the functional equation we have $\psi'\left(\hat{z}^{p^k}\right) = 0$ for $k = 1, 2\ldots$ Now we see $\{\hat{z}^{p^k}\}_{k\in\mathbb{N}}$ forms a sequence of critical points converging to 0, but then by continuity ${\psi_r}'(0) = 0$, which is a contradiction with the Theorem 3.1.

For injectivity we must first have for small $\hat{z}$, we have $|\phi(\psi(\hat{z}))| = |\phi(\psi_r(\hat{z}))| = |\hat{z}|$ and by Theorem 3.2 and using the functional equation we have:

$$|\phi(\psi(\hat{z}))|^{p^k} = \left|\phi(f^{\circ k}(\psi(\hat{z})))\right| = \left|\phi(\psi(\hat{z}^{p^k}))\right| = |\hat{z}|^{p^k}$$

hence we can extend the the identity $|\phi(\psi(\hat{z}))| = |\hat{z}|$ to $\mathbb{D}_\varepsilon$. Now suppose $\psi(\hat{z}_1) = \psi(\hat{z}_2)$, then applying $|\phi|$, we get $|\hat{z}_1| = |\hat{z}_2|$. We must have some minimal value t such that $\exists \hat{z}_1, \hat{z}_2 \in \mathbb{D}_\varepsilon, |\hat{z}_1| = |\hat{z}_2| = t$ and $\psi(\hat{z}_1) = \psi(\hat{z}_2)$, due to injectivity of ψ_r. We then have by ψ being an open mapping (by the Open Mapping Theorem 6.6) for $\hat{z}_1'$ sufficiently close to $\hat{z}_1$, $\exists \hat{z}_2'$, such that $\psi(\hat{z}_1') = \psi(\hat{z}_2')$ and $|\hat{z}_1'| = |\hat{z}_2'| < t$, contradiction. Hence ψ is one-to-one.

Now to see that for the case $\varepsilon = 1$, we have a biholomorphism suppose $U = \psi(\mathbb{D}_1)$. For a contradiction we can assume $U \neq \mathcal{A}^0$, then since $U \subset \mathcal{A}^0$, we must have $z_0 \in \partial U$, with $z_0 \in \mathcal{A}^0$ and since ∂U is contained in $\psi(\partial \mathbb{D}_1)$, we can have a sequence $\psi(\hat{z}_j) \longrightarrow z_0$, where $|\phi(\psi(\hat{z}_j))| \longrightarrow |\phi(z_0)|$, so $|\phi(z_0)| = 1$, this gives a contradiction for the image of $\mathcal{A}$ under $|\phi|$. This also completes the discussion about critical points in this case.

Now when $\varepsilon < 1$, for U as before, we have using the functional equation and continuity:

$$f(U) = f(\psi(\mathbb{D}_\varepsilon)) = \psi(\mathbb{D}_{\varepsilon^n}) \subset \psi(\overline{\mathbb{D}_{\varepsilon^n}}) \subset \overline{\psi(\mathbb{D}_{\varepsilon^n})}$$

so we have $U \subset f^{-1}(\overline{\psi(\mathbb{D}_{\varepsilon^n})}))$. Now the right hand side is closed (continuity of f), we have $\overline{U} \subset f^{-1}(\psi(\overline{\mathbb{D}_{\varepsilon^n}})) \subset \mathcal{A}$, due to connectedness we have $\overline{U} \subset \mathcal{A}^0$. This shows that $\partial U \subset \mathcal{A}^0$.

Now all that is left is to show there's a critical point of f on ∂U. Suppose ∂U contains no critical points of f. Otherwise, let $\hat{z}_0 \in \partial \mathbb{D}_\varepsilon$ and let z_o be an accumulation point of $\psi(\hat{z}_0 t)$, where $t \in [0, 1)$ as $t \longrightarrow 1$. Then by the Inverse Function Theorem we can find open sets W, V such that $z_0 \in W$ and $f(z_0) \in V$, where f is invertible on them with holomorphic inverse f^{-1}. Hence for all $z \in W$ we have $f^{-1}(f(z)) = z$. We can this extend ψ to W, by some neighbourhood of $\hat{z}_0$, by $\psi(\hat{z}) = f^{-1}(\psi(\hat{z}^p))$. Doing this for all points on $\partial \mathbb{D}_\varepsilon$, due to compactness of the boundary, we can increase ε contradicting the minimality of it.

□

Example 3.4. To actually make more sense of our deliberations we will present an example of such a Böttcher map. It doesn't often happen that we can find a closed form expression for these kind of transforms. Take the rational function:

$$f(z) = \frac{z^2}{1 - 2z^2} \approx z^2 + 2z^4 + 4z^6 \ldots$$

with the expansion valid for $|z| < \frac{1}{\sqrt{2}}$. We can see there's a super-attracting fixed point at 0 and furthermore there's no critical points except at $z = 0$. Hence by our Theorem 3.3 the extension of the inverse of the map will be valid in whole of $\mathbb{D}_1$. In this case it is in fact easier to give the inverse ψ, which should satisfy the functional equation $f(\psi(\hat{z})) = \psi(\hat{z}^2)$, since the local degree is 2. Our inverse in this case is given by:

$$\psi(\hat{z}) = \frac{\hat{z}}{1+\hat{z}^2}$$

We then see:

$$f(\psi(\hat{z})) = f\left(\frac{\hat{z}}{1+\hat{z}^2}\right) = \frac{\hat{z}^2}{1+\hat{z}^4} = \psi(\hat{z}^2)$$

3.3 Polynomial Dynamics

Everything we have done so far has been proved for $\mathbb{C}$. However these results can be generalised to the Riemann Sphere $\widehat{\mathbb{C}}$, which will help us understand the behaviour of a polynomial near infinity. More definitions and discussions on the Riemann Sphere can be found in the Appendix.

Let

$$f(z) = a_d z^d + a_{d+1} z^{d+1} \cdots + a_1 z + a_0 \tag{5}$$

where $d \geq 2$ be defined on the Riemann Sphere. We can assume without loss of generality $a_d \neq 0$ in fact by using the conjugation $cf(\frac{z}{c})$, where $c^{d-1} = a_d$ we can get a monic polynomial, so we limit our discussion to this case.

Now we can move on some results helping us understand how such polynomials behave at infinity and how we can apply our previously established results there and in parallel we will get a more concrete idea of the set of elements with bounded orbits for such maps. We formalise this as:

Definition 3.5. We call the set of all $z \in \widehat{\mathbb{C}}$ with a bounded orbit under f the *filled Julia set* of f, $\mathcal{K} = \mathcal{K}(f)$.

Theorem 3.6. *For any polynomial f of degree at least 2, the filled Julia set $\mathcal{K} \subset \widehat{\mathbb{C}}$ is compact, with connected complement and with $\partial\mathcal{K} = \mathcal{J} = \mathcal{J}(f)$ (the Julia set) and with interior equal to the union of all the bounded components U of the Fatou set $\widehat{\mathbb{C}} \setminus \mathcal{J}$. Thus K is equal to the union of all such U and $\mathcal{J}$ itself.*

Proof. We can assume without loss of generality that f is monic as previously discussed. Clearly as $|z| \longrightarrow \infty$, we have $\frac{f(z)}{z^d} \longrightarrow 1$. Hence $\exists r_0 \in \mathbb{R}_{\geq 2}$ such that $\left|\frac{f(z)}{z^d} - 1\right| < \frac{1}{2}, \forall\, |z| < r_0$, then:

$$|f(z)| > \frac{|z^d|}{2} > 2\,|z|$$

By induction it is easy to notice $\left|f^{\circ k}\right| > 2^k r_0$, hence clearly $f^{\circ k}(z) \longrightarrow \infty$.

Let $U = z : |z| > r_0$. Define $\mathcal{A}(f, \infty)$ as the set of all elements with an unbounded orbit under f. Then clearly we have:

$$\mathcal{A}(f, \infty) = \{z : \exists k \in \mathbb{Z}, f^{\circ k}(z) \in U\} = \bigcup_{k=0}^{\infty} \left(f^{-1}\right)^{\circ k}(U)$$

Now since U is open (as a complement of a closed set) and f is continuous, we have $\mathcal{A}(f, \infty)$ is open and hence (as we have $K = \hat{C} \setminus \mathcal{A}$) K is closed. It is also bounded (contained in $\overline{\mathbb{D}_{r_0}}$) and hence it is compact. By Lemma 6.8 we have $\partial\mathcal{K} = \partial\mathcal{A} = \mathcal{J}(f)$.

We must now show that $\mathcal{A} = \mathcal{A}(f, \infty)$ is connected. Let V be any connected component of the Fatou set. By Lemma 6.8 it is either contained in $\mathcal{A}$ or disjoint from it. If V is unbounded then it unique (it contains all of U) and hence it is contained in $\mathcal{A}$. So all that is left is to show any bounded component of the Fatou set is disjoint from $\mathcal{A}$. Let V be such a bounded component. For contradiction suppose $V \subset \mathcal{A}$. Then $\partial V \subset \partial\mathcal{A} \subset K$, so by Maximum Modulus Principle $\left|f^{\circ d}(z)\right| \leq r_0, \forall z \in V$. Hence V is contained in K, contradiction. So $\mathcal{A}$ is the unique unbounded component, hence it is connected and we are done.

□

Now we can further consider the fixed point at ∞ of the polynomial f of degree ≥ 2. Using the same notation as at the start of the section, we can make the substitution $Z = \frac{1}{z}$. And take the rational map (to study it's behaviour at 0):

$$F(\zeta) = \frac{1}{f(1/\zeta)}$$

Then assuming f is monic, near ∞, $f(z) \approx z^d$. By that we have near 0,

$$F(Z) \approx \frac{1}{z^d} = Z^d$$

Hence (as $d \geq 2$) we have a super-attracting fixed point of F at 0. This can be also shown more explicitly using the power series:

$$F(Z) = \frac{z^{-d}}{1 + a_{d-1}z^{-1} + a_{d-2}z^{-2} + \dots a_0 z^{-d}} = \frac{Z^d}{1 + a_{d-1}Z + a_{d-2}Z^2 + \dots a_0 Z^d} =$$

$$= Z^d \sum_{i=0}^{\infty} \left(a_{d-1}Z + a_{d-2}Z^2 + \dots a_0 Z^d\right)^i = Z^d - a_{d-1}Z^{d+1} \dots$$

Then from Theorem 3.1 we can get a map Φ, which conjugates F locally around 0 to the power map $\hat{z} \mapsto \hat{z}^d$. Again by a change of coordinates we get:

$$\phi(z) = \frac{1}{\Phi(\frac{1}{z})}$$

which maps some neighbourhood of ∞ biholomorphically onto another neighbourhood of ∞. We then have:

$$\phi(f(z)) = \phi(z)^d$$

This motivates our next and final theorem, which is also a simple corollary of all that we have done so far:

Corollary 3.7. *Let f be a polynomial of degree ≥ 2. If the filled Julia set $\mathcal{K}$ contains all of the finite critical points of f, the complement of $\mathcal{K}$ is conformally isomorphic to the exterior of the closed unit disc $\overline{\mathbb{D}_1}$ under an isomorphism:*

$$\phi : \widehat{\mathbb{C}} \setminus \mathcal{K} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_1}$$

which conjugates f to the d-th power map. On the other hand if at least one critical point of f belongs to $\mathbb{C} \setminus \mathcal{K}$, then the map ϕ is defined on a subset of $\hat{C} \setminus \mathcal{K}$.

Proof. Bearing the discussion prior to this Corollary in mind, we see that a natural conjugate of Φ, ϕ arises in a neighbourhood of infinity.

Consider first when there are no critical points in the basin $\mathcal{A}(f, \infty)$, then there are no critical points in $\mathcal{A}(F, 0)$ and by connectedness proved in Theorem 3.6 we see that due to Theorem 3.3 the inverse of Φ is defined on $\Psi : \mathbb{D}_1 \to \mathcal{A}(F, 0)$, so naturally we have $\Phi : \mathcal{A}(F, 0) \to \mathbb{D}_1$ and so:

$$\phi : \widehat{\mathbb{C}} \setminus \mathcal{K} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_1}$$

As $\mathcal{A}(f,\infty) = \widehat{\mathbb{C}} \setminus \mathcal{K}$ and $\Phi(Z) \in \mathbb{D}_1 \implies \phi(1/Z) = \frac{1}{\Phi(Z)} \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_1}$.

The other case is done analogically considering Theorem 3.3.

□

Before ending this section we will give a final example with binds all of the results we have seen so far. It is connected to the example we considered before closed form expressions of Böttcher maps aren't easy to come by, so we have to recycle:

Example 3.8. Let's take the map:

$$f(z) = z^2 - 2$$

This has a super-attracting fixed point at ∞. Then notice if we use the same method we previously discussed (substitute $Z = 1/z$ and get map $F(Z)$), we get a map:

$$F(Z) = \frac{1}{f\left(\frac{1}{Z}\right)} = \frac{Z^2}{1 - 2Z^2}$$

which is the exact same map as in the previous example with a fixed point at 0. Hence we can use the ψ analogically to how we discussed finding ϕ. For F we had the local inverse $\Psi(\hat{z}) = \frac{\hat{z}}{1+\hat{z}^2}$, here we have:

$$\psi(\hat{z}) = \frac{1}{\Psi(1/\hat{z})} = \hat{z} + \frac{1}{\hat{z}}$$

which considering the critical points of f will be defined on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_1}$. For verification we find:

$$f\left(\hat{z} + \frac{1}{\hat{z}}\right) = \hat{z}^2 + \frac{1}{\hat{z}^2} = \psi(\hat{z}^2)$$

4 Parabolic Fixed Points

We now consider the case when λ is a root of unity, $\lambda^r = 1$ for $r \in \mathbb{Q}$. In this case, we call $z^* = 0$ a *parabolic fixed point*. If $p+1$ is the smallest integer > 1 such that z^{p+1} has a non-zero coefficient, we express the analytic map f

$$f(z) = \lambda z + \mu z^{p+1} + o(z^{p+1}) \tag{6}$$

In order to characterise the attractive or repulsive nature of the fixed point, we are first interested in finding the *attraction or repulsion directions* to a fixed point.

4.1 The Case $\lambda = 1$

If $\lambda = 1$, then the linear part of f is the identity function and so the function behaves like the identity near the fixed point. In this case, $p + 1$ is called the *multiplicity* of the fixed point, as it is the multiplicity of the $z = 0$ root of $f(z) - z$.

We will first intuitively ponder about the nature of the fixed point in this case, and then formalize our results in Theorem 4.1.

- Let $\varepsilon \in \mathbb{C}$ and consider $f(\varepsilon) = \alpha(\varepsilon)\varepsilon$ where $\alpha(\varepsilon)$ is a positive real function of ε ($\forall\varepsilon, \alpha > 1$ corresponding to repulsion; $\forall\varepsilon, \alpha < 1$ corresponding to attraction). Substituting in Eq. 6 to $p+1$ order in z with $\lambda = 1$:

  $$\varepsilon^p = (\alpha - 1)/\mu$$

  We can see that there are p *attraction vectors*, which correspond to the argument of ε when $\alpha < 1$, and p *repulsion vectors*, which correspond to the argument of ε when $\alpha > 1$.

  $$\begin{aligned} v_-^p &= -1/(p\mu) \\ v_+^p &= +1/(p\mu) \end{aligned} \tag{7}$$

  (The reason for using $1/(p\mu)$ over $1/\mu$ is not important, and will become clear later.) Alternatively we may write, where v_0 is some repulsion vector, that v_j, defined as below, is attractive for odd j and repulsive for even j:

  $$v_j = v_0 \exp(j/p \cdot \pi i) \tag{8}$$

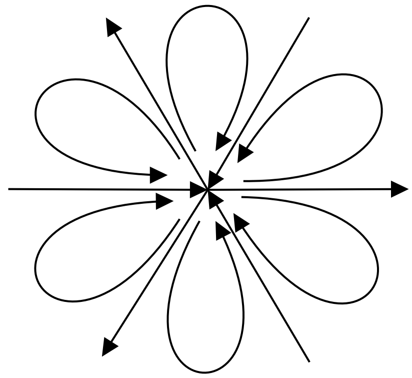

Figure 3: Attraction and repulsion vectors, basins where $p = 3$, $\mu \in \mathbb{R}^{>0}$

- We may also be interested in asking about the *rate* of convergence of the iteration. Since $\lim_{\varepsilon \longrightarrow 0} \alpha(\varepsilon) = 1$, which means the convergence is slower than exponential – in addition, the convergence is slower for larger p. We may guess that the sequence $f^{\circ n}(z)$ converges $\propto v_{-}/n^{1/p}$.
- Taking a cue from the nature of linear dynamical systems, we might imagine the local behaviour of the iterated map to be something like that shown in Fig. 4.1: that the sectors between consecutive repulsion vectors define *attraction basins* for the attraction vector between them – this would also imply that there are no small cycles near a parabolic fixed point.

Theorem 4.1. *Let f be a holomorphic function with a parabolic fixed point at 0 with multiplier $\lambda = 1$. Let $z_0 \in \mathbb{C}$ be such that the sequence $z_n = f^{\circ n}(z_0) \longrightarrow 0$ but $\forall n, z_n \neq 0$. Then, for some attraction vector v_j satisfying $v_j^p = -1/(p\mu)$,*

$$\lim_{n \longrightarrow \infty} n^{1/p} z_n = v_j$$

i.e. $z_n \sim v_j/n^{1/p}$ asymptotically. z_n is said to tend to 0 in the direction of *v_j.*

Proof. The proof relies on a variable substitution $\omega(z) = -1/(p\mu z^p)$, $\tilde{z}_n = \omega(z_n)$. Although the substitution is not injective on all of $\mathbb{C} \setminus \{0\}$, one may define $2p$ restrictions of the function, $\omega_j(z) : \Delta_j \to \mathbb{C} \setminus \mathbb{R}_{(-1)^j}$, where $\Delta_j = \{re^{i\theta} v_j : r > 0, |\theta| < \pi/p\}$. Then consider the map $\tilde{f}_j(\tilde{z}) := \omega_j \circ f \circ {\omega_j}^{-1}(\tilde{z})$, defined for $\tilde{z}$ outside a large disc on $\mathbb{C} \setminus \mathbb{R}_{(-1)^k}$ – whose power series it is easy to compute. For simplicity denoting the pth root in

Δ_j by simply the notation $\sqrt[p]{-}$:

$$f\left(\omega_j^{-1}(\tilde{z})\right) = \sqrt[p]{-\frac{1}{p\mu\tilde{z}}}\left(1 - \frac{1}{p\tilde{z}} + o\left(1/\tilde{z}\right)\right) \tag{9}$$

$$\omega\left(f\left(\omega_j^{-1}(\tilde{z})\right)\right) = \tilde{z}\left(1 - \frac{1}{p\tilde{z}} + o\left(1/\tilde{z}\right)\right)^{-p} \tag{10}$$

$$= \tilde{z}\left(1 + \frac{1}{\tilde{z}} + o\left(1/\tilde{z}\right)\right) \tag{11}$$

$$= \tilde{z} + 1 + o(1) \tag{12}$$

$$\tilde{f}_j(\tilde{z}) = \tilde{z} + 1 + o(1) \tag{13}$$

Then clearly $\tilde{z}_n \sim n$ as $n \longrightarrow \infty$ (formally: $\tilde{z}_{n+1} - \tilde{z}_n \longrightarrow 1$, hence the partial sum $(\tilde{z}_n - \tilde{z}_0)/n \longrightarrow 1$), which implies

$$z_n^p \sim -1/(p\mu n) \tag{14}$$

Further, from Eq. 13, $\exists R > 0, \mathrm{Re}(\tilde{z}) > R \implies \left|\tilde{f}_j(\tilde{z}) - (\tilde{z}+1)\right| < 1/2$. Then $\tilde{\mathcal{P}} := \{\tilde{z} \mid \mathrm{Re}(\tilde{z}) > R\}$ is closed under $\tilde{f}_j$, and the *attracting petal* $\mathcal{P}_j = {\omega_j}^{-1}\left(\tilde{\mathcal{P}}\right)$ is closed under f. Now since $\exists m, \mathrm{Re}(\tilde{z}_m) > R$, we must have $z_m \in \mathcal{P}_j$ for some j, and by closure of the attracting petal, all further z_k are in this petal. Since by definition $\mathcal{P}_j \subseteq \Delta_j$, this means z_n is eventually in Δ_j, and we can take the nth root of Eq. 14:

$$z_n \sim v_j/n^{1/p} \tag{15}$$

□

Corollary 4.2. *Let z_0 be such that the sequence $z_n = \left(f^{-1}\right)^{\circ n}(z_0) \longrightarrow 0$ but $\forall n, z_n \neq 0$. Then, for some repulsion vector v_j of f as defined in Eq. 8, $z_n \sim v_j/n^{1/p}$.*

4.2 The Case $\lambda = \exp(q/r \cdot 2\pi i)$

Once again, consider f as in Eq. 6, but with λ any primitive rth root of unity. Then near its fixed point $z = 0$, $f(z) \approx \lambda z$, which for $\lambda \neq 1$ does not have "true" attraction and repulsion vectors in the sense that we have been imagining them.

Instead, we define attraction and repulsion vectors in a way that is more analogous to the notion of limit points, in that we are satisfied with being asymptotic to subsequences.

Definition 4.3. Let f be a holomorphic function with parabolic fixed point at 0, and v be a complex number.

- If there exists a sequence $z_n = f^{\circ n}(z_0) \longrightarrow 0$ (but $\forall n, z_n \neq 0$) with a subsequence z_{n_k} such that $\arg z_{n_k} \longrightarrow \arg v$, then v is called an attraction vector for f.
- If there exists a sequence $z_n = \left(f^{-1}\right)^{\circ n}(z_0) \longrightarrow 0$ (but $\forall n, z_n \neq 0$) with subsequence z_{n_k} such that $\arg z_{n_k} \longrightarrow \arg v$, then v is called an repulsion vector for f.

Theorem 4.4. *Let f be a holomorphic function with parabolic fixed point at 0, and multiplier $\lambda = \exp(q/r \cdot 2\pi i)$ where q/r is a fraction in its lowest terms. The attraction vectors of f are the same as the same as those of $f^{\circ r}$, and their number is a multiple of r.*

Proof. Any sequence $z_n = f^{\circ n}(z_0) \longrightarrow 0$ can be partitioned into subsequences z_{kr+s} for $s = 0, \ldots r-1$. Each subsequence is an orbit under $f^{\circ r}$, which is a function of the form discussed in IIIa, thus the attraction vectors of f are precisely those of $f^{\circ r}$. For each v asymptotic to z_{kr}, $\lambda^s v$ is asymptotic to z_{kr+s}. Thus any $z_n \longrightarrow 0$ gives rise to r attraction vectors for f. □

Corollary 4.5. *The multiplicity of the fixed point at 0 of $f^{\circ r}$ is congruent to 1 mod r.*

Theorem 4.4 is crucial, as it allows us to often reduce problems about parabolic points to the case IIIa. We will be making this without loss of generality assumption about f in sections that follow, and the general case will follow easily through Theorem 4.4.

4.3 Petals and Basins

As we have seen, unlike with attracting and repelling fixed points, parabolic fixed points act in a manner that is simultaneously both attracting and repelling. Thus the notion of an attraction basin in Definition 2.6 will for the most part be replaced for our purposes by a more specialized Definition 4.6. Similarly, open neighbourhoods of the origin will often be substituted by the notion of a *petal* per 4.7 (although the analogy isn't exact: a superset of a petal is not necessarily a petal).

Definition 4.6. Let v be an attraction vector at fixed point 0. The basin of attraction $\mathcal{A}_v$ for v is defined as the set of points z such that $f^{\circ n}(z) \longrightarrow 0$ in the direction of v. The immediate basin of attraction $\mathcal{A}_v^0$ is defined as the unique connected component of $\mathcal{A}_v$ that is closed under f.

Definition 4.7. Let f be a holomorphic function with parabolic fixed point. Where f is injective on some neighbourhood $\mathcal{N}$ of its fixed point, an open set $\mathcal{P} \subseteq \mathcal{N}$ is called an attracting petal for f along attraction vector v if

1. $\mathcal{P}$ is closed under f.
2. $\mathcal{P} \subseteq \mathcal{A}_v$
3. Any orbit $f^{\circ n}(z_0)$ converging to 0 along v is eventually in $\mathcal{P}$.

Lemma 4.8. *An attraction basin $\mathcal{A}_v$ is open.*

Proof. We have already seen an example of a petal $\mathcal{P}$ from the proof of Theorem 4.1 (and one can verify it satisfies the properties). By definition, the orbit of any element z in the attraction basin is eventually in some such $\mathcal{P}$, i.e. $f^{\circ n}(z) \in \mathcal{P}$ for sufficiently large n. One may construct a sufficiently small open neighbourhood around $f^{\circ n}(z)$, and take its preimage under $f^{\circ n}$ – by continuity of $f^{\circ n}$, this preimage must be an open neighbourhood around z. □

The following lemma is analogous to Theorem 6.8.

Lemma 4.9. *For f a holomorphic function with parabolic fixed point, the basins of attraction $\mathcal{A}_v$ are contained in the Fatou set of f, while their boundaries $\partial\mathcal{A}_v$ are contained in the Julia set.*

Proof. By Lemma 4.8, $\mathcal{A}_v$ does not contain its boundary. Thus any $z \in \partial\mathcal{A}_v$ is not in $\mathcal{A}_v$. So either:

- $f^{\circ n}(z)$ trivially converges to the fixed point, i.e. is eventually 0 (which is in the Julia set due to its proximity to both attracting and repelling orbits).
- $f^{\circ n}(z)$ does not converge to the fixed point, but is close to points that do, and is therefore in the Julia set.

□

We easily arrive at the repulsive version of Definition 4.7 by replacing f with f^{-1}.

Somewhat "better" petals than the ones used in Theorem 4.1 are formalized in the following theorem:

Theorem 4.10 (Parabolic flower theorem)**.** *Let f be a holomorphic function as in Eq. 6 with $\lambda = 1$. In any neighbourhood of the fixed point 0, there exist simply connected petals $\mathcal{P}_0 \dots \mathcal{P}_{2p-1}$ (even subscripts repulsive, odd subscripts attractive) such that:*

- *Their union is a punctured open neighbourhood of 0.*

- *Any two non-adjacent petals are disjoint.*
- *Each $\mathcal{P}_j$ has a simply-connected region of intersection with $\mathcal{P}_{j+1}$ and another simply-connected region of intersection with $\mathcal{P}_{j-1}$.*

(When $p+1=2$, the right- and left- neighbours are the same, but there are still two simply-connected regions of intersection.)

Proof. Recall the substitution ω, and the defined quantity R, in the proof of Theorem 4.1. Then define $\tilde{\mathcal{P}}^- := \{x+iy \mid x+|y| > 2R\}$ and $\tilde{\mathcal{P}}^+ := -\tilde{\mathcal{P}}^-$. Then define:

$$\mathcal{P}_j = \begin{cases} {\omega_j}^{-1}(\tilde{\mathcal{P}}^+) & j \text{ even} \\ {\omega_j}^{-1}(\tilde{\mathcal{P}}^-) & j \text{ odd} \end{cases}$$

One can check that the even and odd petals are indeed repulsive and attractive respectively. Each of the required statements can be verified rather easily:

- Since $\tilde{\mathcal{P}}^+ \cup \tilde{\mathcal{P}}^-$ contains all complex numbers with a radius over $2R$, each petal $\mathcal{P}_j$ contains a sector centered at the fixed point, of angle spanning Δ_j and radius $(2Rp\mu)^{-1/p}$, Thus their union covers all such sectors, i.e. an open disc.
- Non-adjacent Δ_j are disjoint, and each $\mathcal{P}_j \subseteq \Delta_j$.
- $\tilde{\mathcal{P}}^+ \cap \tilde{\mathcal{P}}^- = \tilde{\mathcal{Q}}^\vee \cap \tilde{\mathcal{Q}}^\wedge$ where the right-hand-side is a disjoint union, $\mathcal{Q}^\vee = \{x+iy \mid y-|x| > 2R\}$ and $\mathcal{Q}^\wedge = -\mathcal{Q}^\vee$. These regions correspond to the intersections of $\mathcal{P}_j$ with each of its neighbouring petals, i.e. $\omega(\mathcal{P}_j \cap \mathcal{P}_{j+1})$ is either $\mathcal{Q}^\vee$ or $\mathcal{Q}^\wedge$ depending on the parity of j.

□

4.4 Abel Linearisation

We now start to think about constructing a linearisation for a holomorphic function near a parabolic fixed point. Our earlier experience with the Koenigs linearisation might suggest a linearisation of the form $\hat{f}(\hat{z}) = \hat{z}$, but this is obviously absurd: it would require identifying points that ought not be identified.

Instead, we are inspired by the structure of a petal, which comes with some notion of "direction" defined on it. One may imagine rearranging the petal so that application of f is just adding 1, i.e. consider (where $\phi(z) = \hat{z}$,

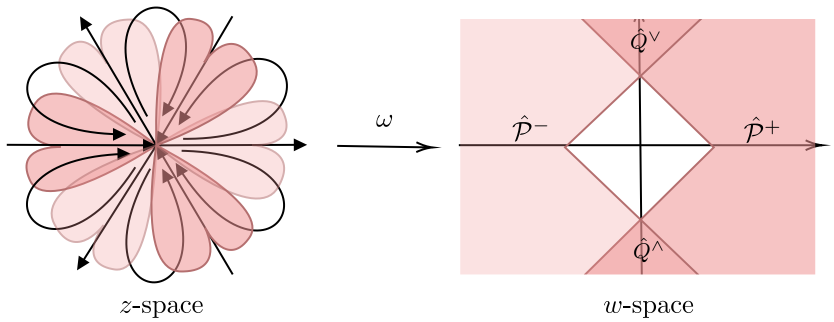

Figure 4: Illustration of the petals constructed in Theorem 4.10.

$\hat{f} = \phi \circ f \circ \phi^{-1}$) a linearisation of the form $\hat{f}(\hat{z}) = \hat{z} + 1$. One may imagine that this would be closely related to the quotient $\mathcal{P}/f$.

Theorem 4.11 (Parabolic linearisation theorem)**.** *Let $\mathcal{P}$ be a petal for holomorphic function f at a parabolic fixed point. There exists a unique (up to composition on the left with translation) conformal embedding $\phi : \mathcal{P} \to \mathbb{C}$ called a Fatou co-ordinate on $\mathcal{P}$ such that, for all $z \in \mathcal{P} \cup f^{-1}(\mathcal{P})$, we have:*

$$\phi(f(z)) = \phi(z) + 1$$

Here's an idea to construct this ϕ: recall how in $\tilde{z}$-space, we have $\tilde{f}(\tilde{z}) = \tilde{z} + 1 + o(1)$. Well, it is not exact, but as you keep applying $\tilde{f}$, $|\tilde{z}| \longrightarrow \infty$ and it becomes more and more exact. So we may consider representing $\tilde{z}$ by some $\tilde{f}^{\circ n}(\tilde{z})$ for large n. More precisely, we'd need to set some base point and replace $\tilde{z} \mapsto \tilde{f}^{\circ n}(\tilde{z}) - \tilde{f}^{\circ n}(\tilde{z}_O)$.

Lemma 4.12. *Where $\tilde{\mathcal{P}}_R = \{\tilde{z} \mid \mathrm{Re}(\tilde{z}) > R\}$ for some R, and $\tilde{f} : \tilde{\mathcal{P}}_R \longrightarrow \tilde{\mathcal{P}}_R$ is an injective holomorphic function satisfying the following inequalities, for constants $c, \varepsilon > 0$:*

$$\mathrm{Re}(\tilde{f}(\tilde{z})) > \mathrm{Re}(\tilde{z}) + 1/2$$

$$\left|\tilde{f}(\tilde{z}+1) - (\tilde{z}+1)\right| \leq c/\left|\tilde{z}\right|^{\varepsilon}$$

Then, where $\tilde{z}_O$ is some arbitrary base point in $\tilde{\mathcal{P}}_R$, the following sequence of functions:

$$\tilde{\phi}_n(\tilde{z}) = \tilde{f}^{\circ n}(\tilde{z}) - \tilde{f}^{\circ n}(\tilde{z}_O)$$

Converges locally uniformly to a biholomorphic map $\tilde{\phi} : \tilde{\mathcal{P}}_R \to U \subset \mathbb{C}$ that satisfies $\phi(\tilde{f}(\tilde{z})) = \phi(\tilde{z}) + 1$.

Proof. The ratio $\tilde{\phi}_n(\tilde{z})/\tilde{\phi}_{n-1}(\tilde{z})$ is relevant for questions of convergence, and is seen to be the average slope of $\tilde{f}$ along the line segment from $\tilde{f}^{\circ n}(\tilde{z}_O)$ to $\tilde{f}^{\circ n}(\tilde{z})$. Well, when $|\tilde{z}| \geq 2S \geq 2R$, the function $\tilde{f}(\tilde{z}) - (1 + \tilde{z})$ transforms $\mathbb{D}_S(\tilde{z}) \mapsto \mathbb{D}_{c/S^\varepsilon}(0)$, so by the Cauchy derivative estimate (Theorem 6.5):

$$\left|\tilde{f}'(\tilde{z}) - 1\right| < c/S^{1+\varepsilon}$$

The same bound thus applies on the average slope between $\tilde{z}_1, \tilde{z}_2 \in \tilde{\mathcal{P}}_{2S}$:

$$\left|\frac{\tilde{f}(\tilde{z}_2) - \tilde{f}(\tilde{z}_1)}{\tilde{z}_2 - \tilde{z}_1} - 1\right| \leq \frac{c}{S^{1+\varepsilon}}$$

Since $\mathrm{Re}(\tilde{f}^{\circ n}(\tilde{z})) > n/2$, we can write for all $n \geq 1$, $c' = 2^{1+\varepsilon}c$:

$$\left|\frac{\tilde{\phi}_n(\tilde{z})}{\tilde{\phi}_{n-1}(\tilde{z})} - 1\right| \leq \frac{c'}{n^{1+\varepsilon}} \tag{16}$$

$$1 - \frac{c'}{n^{1+\varepsilon}} \leq \left|\frac{\tilde{\phi}_n(\tilde{z})}{\tilde{\phi}_{n-1}(\tilde{z})}\right| \leq 1 + \frac{c'}{n^{1+\varepsilon}}$$

Noting that $K = \prod \left(1 + c'/n^{1+\varepsilon}\right)$ is finite:

$$\left|\tilde{\phi}_n(\tilde{z})\right| \leq K\ |\tilde{z} - \tilde{z}_O| \tag{17}$$

Considering Eq. 17 for $n \mapsto n - 1$ and multiplying by Eq. 16,

$$\left|\tilde{\phi}_n(\tilde{z}) - \tilde{\phi}_{n-1}(\tilde{z})\right| \leq Kc' \left|\tilde{z} - \tilde{z}_O\right| / n^{1+\varepsilon} \tag{18}$$

If we sum Eq. 18, it is clear that the sum of the right-hand-side converges absolutely, thus the sum:

$$\tilde{\phi}_0(\tilde{z}) + \sum_{n=1}^{\infty} \left(\tilde{\phi}_n(\tilde{z}) - \tilde{\phi}_{n-1}(\tilde{z})\right)$$

Is absolutely convergent, therefore the desired limit exists $\forall \tilde{z} \in \tilde{\mathcal{P}}_{2R}$:

$$\tilde{\phi}(\tilde{z}) = \lim_{n \longrightarrow \infty} \tilde{\phi}_n(\tilde{z})$$

From Eq. 18, we see that $\tilde{\phi}_n(\tilde{z})/\left|\tilde{z} - \tilde{z}_O\right|$ converges uniformly to $\tilde{\phi}(\tilde{z})/\left|\tilde{z} - \tilde{z}_O\right|$ (because the error does not depend on $\tilde{z}$), which shows that the limiting function $\phi(\tilde{z})$ is holomorphic, and the injectivity of ϕ follows immediately from the injectivity of f and injectivity of the uniform limit of injective functions. $\square$

Proof of Theorem 4.11 – Existence. In the case where the petal is $\mathcal{P}_R = \omega^{-1}(\tilde{\mathcal{P}}_R)$, the function $\phi = \tilde{\phi} \circ \omega$ suffices, with $\tilde{\phi}$ defined as in Lemma 4.12 and ω defined as in 4.1. For an arbitrary petal $\mathcal{P}$: recall that by definition of a petal, any $z \in \mathcal{P}$ must eventually have $f^{\circ n}(z) \in \mathcal{P}_R$ – so define $\phi(z) = \tilde{\phi} \circ \omega \circ f^{\circ n}(z) - n$. $\square$

For uniqueness, we will need a minor lemma (which should be read after).

Lemma 4.13. *The union of all integer translations of U as defined in Lemma 4.12, $U + \mathbb{Z}$, is all of $\mathbb{C}$.*

Proof. We wish to show that $\forall \tilde{z} \in \mathbb{C}, \exists m \in \mathbb{Z}, \tilde{z} + m \in U$. Where S is large enough that $\forall \tilde{z}. \in \tilde{\mathcal{P}}_R \cap (\overline{\mathbb{D}}_S(0))^c, \left|\tilde{\phi}(\tilde{z}.) - \tilde{z}.\right| < |\tilde{z}.|/3$, choose a $\tilde{z}. = \tilde{z} + m$ with a sufficiently high real part that $|\tilde{z}.| > 2S$ and $\overline{\mathbb{D}}_{|\tilde{z}.|/2}(\tilde{z}.) \subset \tilde{\mathcal{P}}_R$. Then for any $\tilde{z}_\bullet \in \overline{\mathbb{D}}_{|\tilde{z}.|/2}(\tilde{z}.)$, we have $|\tilde{z}_\bullet| > S \implies \left|\tilde{\phi}(\tilde{z}_\bullet) - \tilde{z}_\bullet\right| < |\tilde{z}_\bullet|/3 < (3|\tilde{z}.|/2)/3 = |\tilde{z}.|/2$. Since $|\tilde{z}.|/2$ is the radius of $\overline{\mathbb{D}}_{|\tilde{z}.|/2}(\tilde{z}.)$, we have that for all $\tilde{z}_\circ \in \partial\overline{\mathbb{D}}_{|\tilde{z}.|/2}(\tilde{z}.), \left|\tilde{\phi}(\tilde{z}_\circ) - \tilde{z}_\circ\right| < |\tilde{z}_\circ - \tilde{z}.|$. Then by Rouche's theorem, the function $\tilde{\phi}(\tilde{z}_\bullet) - \tilde{z}. = (\tilde{\phi}(\tilde{z}_\bullet) - \tilde{z}_\bullet) + (\tilde{z}_\bullet - \tilde{z}.)$ has the same number of zeroes as $\tilde{z}_\bullet - \tilde{z}.$, i.e. one, i.e. $\exists \tilde{z}_\bullet$ such that $\tilde{\phi}(\tilde{z}_\bullet) = \tilde{z}.$. Hence $\tilde{z}. \in U$. $\square$

Proof of Theorem 4.11 – Uniqueness. First consider the case of $\mathcal{P}_R$, as before, and consider some alternative Abel linearisation $\phi' : \mathcal{P}_R \to U'$. Then $E = \phi' \circ \phi^{-1}$ is a bijection $U \to U'$ that preserves the "plus one" structure, i.e. $E(\tilde{z} + 1) = E(\tilde{z}) + 1$. As the union of integer translations of U is all of $\mathbb{C}$, then we could use this property (i.e. through $E(\tilde{z} + n) = E(\tilde{z}) + n$) to define a a bijective map $E : \mathbb{C} \to U' + \mathbb{Z}$. Such a map must be affine, and the only affine maps satisfying $E(\tilde{z} + 1) = E(\tilde{z}) + 1$ are translations. $\square$

Corollary 4.14 (Cylinder theorem)**.** *For any petal $\mathcal{P}$ attracting or repelling, $\mathcal{P}/f$ is conformally isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$.*

As with the Koenigs linearisation, we can also define global linearisations. However, these are not necessarily injective.

Corollary 4.15 (Global linearisation – attracting petal)**.** *Where $\mathcal{P}$ is an attracting petal in the attracting basin $\mathcal{A}$, the Fatou co-ordinate $\phi : \mathcal{P} \to \mathbb{C}$ extends uniquely to a map $\mathcal{A} \to \mathbb{C}$, still satisfying $\phi(f(z)) = \phi(z) + 1$.*

Corollary 4.16 (Global linearisation – repelling petal)**.** *Where $\mathcal{P}$ is a repelling petal, the inverse map $\phi^{-1} : \phi(\mathcal{P}) \to \mathcal{P}$ extends uniquely to a map $\mathbb{C} \to \mathbb{C}$, satisfying $f(\phi^{-1}(\tilde{z})) = \phi^{-1}(\tilde{z} + 1)$.*

For completeness, we include a result analogous to Lemma 2.8 regarding a maximal disk for the local inverse of ϕ.

Lemma 4.17. *Where f is a non-linear rational map with parabolic fixed point 0 and multiplier $\lambda = 1$:*

1. *each immediate basin of 0 contains at least one critical point of f.*
2. *each basin contains exactly one petal $\mathcal{P}_{\max}$ that maps injectively onto some right half-plane under ϕ that is maximal with respect to this property.*
3. *$\mathcal{P}_{\max}$ has at least one critical point of f on its boundary.*

Proof. We will omit the details of the proof, as it is completely analogous to Lemma 2.8. On some chosen attraction basin $\mathcal{A}$, we consider the local inverse ψ_ε from the inverse function theorem – this is necessarily defined on a domain that contains some right half-plane $\hat{P}_\varepsilon$. We attempt to extend this map via analytic continuation, leftward along horizontal lines in the space of the linear co-ordinate. Such an extension meets an obstruction, implying the existence of a maximal right half-plane on which it is defined (mapped under ψ to a maximal petal $\mathcal{P}_{\max}$). As before, this obstruction implies f failing to be injective, and thus having a critical point. □

4.5 Remarks on the Normal Form

In Sections 2 and 3, the classification of local dynamics up to conjugacy classes was fairly straightforward: the Koenigs linearisation and Böttcher's theorem provided clear and simple algorithms to calculate the normal form of an analytic function near a fixed point. On the other hand, the Abel linearisation is only defined on a single petal.

There is indeed a classification of the conjugacy classes in the parabolic case, known as the Ecalle-Voronin classification, which vaguely relies on "pasting together" the Abel linearisations on each of the $2p$ petals. However, it is rather advanced, having only been discovered as recently as 1981 – we will refer to (Écalle 1981) and (Iliashenko and Yakovenko 2008) for an overview of this topic.

5 Cremer Points and Siegel Discs

5.1 Motivation

Hitherto, we have considered functions with either attracting, geometrically attracting, repelling or rationally indifferent fixed points. We now focus on the case when the fixed point is irrationally indifferent.

Once again, we study maps of the form $f : \mathcal{N} \to \mathbb{C}$

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots \tag{19}$$

with $\mathcal{N} \subseteq \mathbb{C}$ some neighbourhood of the origin and where the origin is a fixed point with multiplier $\lambda = e^{2\pi i \xi}$, $\xi \in \mathbb{R}/\mathbb{Z}$ irrational. We wish to generalise Koenigs linearisation Theorem to the above case. We first formally define the discussion of local linearisation from Chapter 2.

Definition 5.1 (Locally linearisable)**.** The function f above is said to be *locally linearisable* if there is a local biholomorphic map ψ which conjugates f to a linear map:

$$\left(\psi^{-1} \circ f \circ \psi\right)(z) = \lambda z, \tag{20}$$

for all z in some neighbourhood of the origin.

Here, equation (20) is called *Schröder's equation.*

In the special case that f is a globally defined rational function, we have the following lemma:

Lemma 5.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function of degree ≥ 2. Suppose $z_0 \in \mathbb{C}$ is an indifferent fixed point, $\lambda = |f'(z_0)| = 1$. Then the following are equivalent:*

1. *f is locally linearisable around z_0.*
2. *z_0 is in the Fatou set $\mathbb{C} \setminus J(f)$.*
3. *The connected component U of the Fatou set containing z_0 is conformally isomorphic to $\mathbb{D}$, and the isomorphism conjugates f to multiplication by λ on $\mathbb{D}$.*

Proof. Note first that since the conformal isomorphism in (3), if it exists, locally linearises f; so (3) is a strictly stronger statement than (1). It remains only to prove two other implications:

- $1 \implies 2$. Suppose f is locally linearisable: there is a map ϕ univalent in some neighbourhood of z_0 so that $\phi \circ f \circ \phi^{-1}$ is the linear map $w \mapsto \lambda w$ for all w in some disc $\mathbb{D}_r$ around the origin. See that the family $\{f^n\}$ of iterates of f is normal on some neighbourhood of z_0 only if the family $\{\phi \circ f^n \phi^{-1}\}$ is, since composition is continuous on the topology of locally uniform convergence. But $\{\phi \circ f^n \circ \phi^{-1}\}$ is the family of iterates $\{w \mapsto \lambda^n w\}$. By compactness of the unit circle, we extract a convergent subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ so that $\lambda_{n_k} \longrightarrow \lambda$ as $k \longrightarrow \infty$. Together with the bound $|\lambda_{n_k} w - \lambda w| < r\,|\lambda_{n_k} - \lambda|$ we see that the sequence $\{w \mapsto \lambda_{n_k} w\}$ of maps converges uniformly (and hence locally uniformly) to $w \mapsto \lambda w$.
- $2 \implies 3$. This follows from the classification of Fatou components (Theorem 6.9 in the Appendix, whose proof we do not present), considering U as a Riemann surface and seeing that f maps U into itself by the connectedness of U. That U contains an indifferent fixed point excludes the *attracting* and *escape* cases (since the orbit of the fixed point accumulates at itself and does not converge to an attracting fixed point); that the degree of f is no less than 2 prohibits any iterate of f from being the identity on any open set, which excludes the *finite order* case. Finally, U must be conformally isomorphic to a disc since irrational rotations on an annulus or a punctured disc do not have fixed points.

□

With a local linearisation, one can effectively deduce the behaviour of orbits as has previously been discussed. It is thus important to understand when such linearisations exist and what may prevent such an existence.

5.2 Cremer's Nonlinearisation Theorem

Definition 5.3. We say an irrationally indifferent fixed point is a *Cremer point* if there is no local linearisation of f around the fixed point. A connected component of the Fatou set on which f is conjugate to a rotation of the unit disc is called a *Siegel disc*.

We will prove the existence of Siegel discs in section 5.4. We will now study the existence of Cremer points.

In order to answer when local linearisation exists, one can consider the implications of such a linearisation. Suppose we have f and ϕ as in equation (20) so f is conjugate to the linear map $w \mapsto \lambda w$ in a neighbourhood of the origin. An immediate implication is that zero is the only periodic point of the linear map in this neighbourhood and so, as the number of periodic points is invariant under conjugation, we have that zero is an *isolated* periodic point. That is, we have a neighbourhood about 0 in which it is the only periodic point of f.

Thus, if we argue in a contrapositive fashion, if f has periodic points arbitrarily close to the origin, the origin must be a Cremer point. This motivates the following lemma.

Lemma 5.4. *Let $\lambda \in \mathbb{C}$ s.t. $|\lambda| = 1$ and f be a monic polynomial of the form*

$$f(z) = z^d + \cdots + \lambda z,$$

where $d \geq 2$. Suppose that the sequence $\sqrt[d^q]{1/|\lambda^q - 1|}$ is unbounded as $q \longrightarrow \infty$. Then f is not locally linearisable about the origin.

Proof. Certainly $z = 0$ is a fixed point of multiplier λ. For $q \in \mathbb{N}$, $f^{oq}(z)$ is of the form $z^{d^q} + \cdots + \lambda^q z$. Hence, the fixed points of f^{oq}, which correspond to periodic points of f, satisfy the polynomial equation

$$z^{d^q} + \cdots + (\lambda^q - 1)z = 0$$

Denote by $z_q(1), z_q(2), \ldots, z_q(d^q - 1)$ the non-zero roots of this polynomial. Then

$$\prod_{j=1}^{d^q-1} |z_q(j)| = |\lambda^q - 1|$$

If $|\lambda^q - 1| < 1$, $\exists\ j_q$ s.t. $0 < |z_q(j_q)|^{d^q} \leq |z_q(j_q)|^{d^q-1} \leq |\lambda^q - 1| < 1$ (i.e. take $z_q(j_q) = \arg\min_j\{|z_q(j)|\}$). Then

$$0 < |z_q(j_q)| < |\lambda^q - 1|^{1/d^q} \tag{21}$$

Now, if $\sqrt[d^q]{1/|\lambda^q - 1|}$ is unbounded, we can construct a sequence $(q_k)_{k\geq 1}$ so that

$$\begin{aligned} |\lambda^{q_k} - 1|^{-1/d^{q_k}} &\longrightarrow \infty \\ \implies |\lambda^{q_k} - 1|^{1/d^{q_k}} &\longrightarrow 0 \end{aligned}$$

as $k \longrightarrow \infty$. We thus have $K \in \mathbb{N}$ where $|\lambda^{q_k} - 1| < 1$ whenever $k \geq K$ and so from (21) it follows $z_{q_k}(j_{q_k}) \longrightarrow 0$. Hence, every neighbourhood of the origin contains infinitely many periodic points. The result thus follows. □

In fact, one can generalise the above result which leads us to Cremer's proof of the existence of Cremer points.

Theorem 5.5. *(Cremer, 1938) Given $\lambda \in \mathbb{C}$ on the unit circle and $d \geq 2$, if the sequence $\sqrt[d^q]{1/|\lambda^q - 1|}$*

is unbounded as $q \longrightarrow \infty$, no fixed point with multiplier λ of a rational function of degree d can be locally linearisable.

Proof. For the general case, we take a rational function and reduce the problem to that described in the above lemma. Let $f(z) = P(z)/Q(z)$ for P, Q polynomials with no common factors. Let z be a fixed point of f with multiplier λ. Through a conjugation by a Möbius transformation, we can assume that $z = 0$ and so $f(0) = 0 \implies P(0) = 0$.

Claim. $\exists z_1 \neq 0$ such that $f(z_1) = 0$

Proof of claim. Any zero of P will be a zero of f and if $\deg(Q) > \deg(P)$, $z_1 = \infty$ will be a zero of f. The only non-trivial case is hence when $P(z) = z^d$ and $\deg(Q) \leq d$. We also have $|\lambda| = |f'(0)| = 1$. From this, along with $d \geq 2$ and P, Q having no roots in common, it follows that f cannot be of this form. □

By conjugating with another Möbius transformation that takes z_1 to ∞, we can assume that $f(\infty) = f(0) = 0$. Then, we have $d = \deg(Q) > \deg(P)$.

Suppose $Q(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$ where $a_d, a_0 \neq 0$ (as Q and P share no roots and $\deg(Q) = d$). If we conjugate f by the map $z \mapsto (a_0/a_d)^{1/d} z$, we can assume $a_0 = a_d = 1$ so P and Q are of the form:

$$Q(z) = z^d + a_{d-1} z^{d-1} + \cdots + 1, \quad P(z) = \beta_{d-1} z^{d-1} + \cdots + \beta_2 z^2 + \lambda z$$

A brief computation gives

$$f^{oq}(z) = \frac{P_q(z)}{Q_q(z)} = \frac{* z^{d^q - 1} + \cdots + * z^2 + \lambda^q z}{z^{d^q} + \cdots + 1}.$$

The formula for fixed points of f^{oq} is then given by

$$P_q(z) - z Q_q(z) = 0$$

$$\implies z(z^{d^q} + \cdots + (1 - \lambda^q)) = 0$$

. The result now immediately follows from the above lemma. □

Cremer's Theorem gives a seemingly broad class of irrational numbers $\xi \in \mathbb{R}/\mathbb{Z}$ for which there exists a function with multiplier $\lambda = e^{2\pi i \xi}$ that is not locally linearisable. In actual fact, this property holds for a *generic* class of irrational numbers.

Definition 5.6. Let p be a property of angles in $\mathbb{R}/\mathbb{Z}$ and define

$$T = \{\xi \in \mathbb{R}/\mathbb{Z} \mid \xi \text{ satisfies property } p\}$$

Then we say p holds for *generic* ξ if $\exists\ \{U_n\}_{n\in\mathbb{N}}$ dense open subsets of $\mathbb{R}/\mathbb{Z}$ such that

$$\bigcap_{n=1}^{\infty} U_n \subseteq T.$$

By Baire's Theorem, such a countable intersection is necessarily dense and uncountably infinite.

Intuitively, if a property is generic, it may not hold for all points but if we perturb a given point slightly, we would expect to find a point satisfying the property.

Corollary 5.7. *For a generic choice of rotation number $\xi \in \mathbb{R}/\mathbb{Z}$, for any rational function of degree ≥ 2 z_0 with a fixed point of multiplier $e^{2\pi i\xi}$, there is no locally linearising map about z_0*

Proof. (This was problem 11-b in Milnor, 2006).
Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$ be an arbitrary decreasing sequence of positive real numbers converging to 0. Viewing $\mathbb{R}/\mathbb{Z}$ as the interval $[0, 1)$, define

$$S(q_0) = \left\{ \xi \in [0,1) \;\middle|\; \exists\ \frac{p}{q} \in \mathbb{Q} \cap [0,1)\ , q > q_0 \text{ in lowest terms s.t.} \left|\xi - \frac{p}{q}\right| < \varepsilon_q \right\}.$$

Then $S(q_0)$ is a union of open balls so is open and contains all but finitely many rational numbers in [0,1) so is dense in [0,1). Then

$$S = \bigcap_{q_0 \geq 1} S(q_0)$$

is a countable intersection of dense open sets and consists of all angles $\xi \in \mathbb{R}/\mathbb{Z}$ for which there are infinitely many rationals satisfying the defining condition of the sets $S(q_0)$. For $d \geq 2$, let $\varepsilon_q = \frac{1}{2\pi} q^{-(d^q+1)}$. Since, $0 < \varepsilon_q < \frac{1}{q^q}$, the ε_q converge to 0 and form a strictly decreasing sequence. Suppose $\xi \in S$. Now for any $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$, we have

$$|\lambda^q - 1| = \left|e^{2\pi i q\xi} - 1\right| = |2\sin(\pi q\xi)| = |2\sin(\pi(q\xi - p))| \leq 2\pi\,|q\xi - p|\,.$$

Since $\xi \in S$, we have a sequence of rational numbers $\frac{p_i}{q_i}$ such that $q_i \longrightarrow \infty$ and so $|\lambda^{q_i} - 1| \leq 2\pi q_i \left|\xi - \frac{p_i}{q_i}\right| < 2\pi q_i \varepsilon_{q_i}$. Hence $|\lambda^{q_i} - 1|^{1/d^{q_i}} < \frac{1}{q_i} \longrightarrow 0$ as $i \longrightarrow \infty$. Thus $\xi \in S \Longrightarrow \liminf |\lambda^q - 1|^{1/d_q} = 0$. The result thus follows. □

The question still remains whether non-linearisation is the case for *all* irrational rotations. In fact, counter-intuitively, quite the opposite is true and it will turn out that the possibility of a local linearisation depends very carefully on to what extent the irrational angle ξ can be approximated by rational numbers.

5.3 Siegel's Linearisation Theorem

We introduce various classes of irrational numbers and results from number theory in order to understand classical theorems on linearisation as well as sharper, recent results.

Definition 5.8. For $\xi \in \mathbb{R}/\mathbb{Z}$ an irrational number ξ is called *Diophantine of order* $\leq \kappa$ if $\exists\, \varepsilon > 0$ such that

$$\left|\xi - \frac{p}{q}\right| > \frac{\varepsilon}{q^{\kappa}}, \quad \forall\, p/q \in \mathbb{Q}.$$

The set of all κ order Diophantine numbers is denoted by $\mathcal{D}(\kappa)$ so certainly $\mathcal{D}(\kappa) \subseteq \mathcal{D}(\eta)$ whenever $\kappa \leq \eta$

Intuitively, ξ is Diophantine if it is badly approximated by rational numbers.

Lemma 5.9. *Let ξ be irrational and $\lambda = e^{2\pi i \xi}$. Then*

1. $\xi \in \mathcal{D}(\kappa) \Longleftrightarrow \exists M > 0$ *s.t.* $\forall q \in \mathbb{N},\ \left|\lambda^q - 1\right|^{-1} \leq M q^{\kappa - 1}$
2. $\mathcal{D}(\kappa) = \emptyset$ *for* $\kappa = 0, 1$

Proof. 1. Fix $q \neq 0 \in \mathbb{N}$ and let p be the closest integer to $q\xi$ so that $|q\xi - p| \leq \frac{1}{2}$. From the proof of Corollary 5.7, we have

$$|\lambda^q - 1| \leq 2\pi\, |q\xi - p|\,.$$

Moreover, since $|q\xi - p| \leq \frac{1}{2}$, we have $\frac{2}{\pi}\,|\pi(q\xi - p)| \leq |\sin(\pi(q\xi - p))|$ and so

$$4q\left|\xi - \frac{p}{q}\right| \leq |\lambda^q - 1| \leq 2\pi q \left|\xi - \frac{p}{q}\right| \tag{22}$$

If $\xi \in \mathcal{D}(\kappa)$, then immediately from the left inequality of (22), we have

$$\left|\lambda^q - 1\right|^{-1} \leq \frac{1}{4\varepsilon} q^{\kappa - 1}.$$

Conversely, now using the right inequality in (22), if $|\lambda^q - 1|^{-1} \leq Mq^{\kappa-1}$, we have

$$|q\xi - p| \geq \frac{1}{2\pi M q^{\kappa-1}}$$

where $p = \lfloor q\xi \rfloor$. Thus, if $a \in \mathbb{Z}$ is any integer, we have $|q\xi - a| \geq |q\xi - p|$ and so

$$\left|\xi - \frac{a}{q}\right| \geq \frac{\varepsilon}{q^\kappa}$$

and the result follows since $q \in \mathbb{N}$ was arbitrary.

2. (This is problem 11-a in Milnor, 2006.) We prove that for any irrational x there are infinitely many rationals p/q such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}$$

For any integer $Q > 1$, the circle can be partitioned into a disjoint union of Q half-open intervals of length $1/Q$:

$$\mathbb{R}/\mathbb{Z} = \bigcup_{k=0}^{Q-1} \left[\frac{k}{Q}, \frac{k+1}{Q}\right)$$

By the pigeonhole principle, at least two of the $Q+1$ numbers $0, x, 2x, \cdots, Qx$ fall in the same interval in the quotient. That is, there exist integers p and $0 \leq m < n \leq Q$ such that

$$nx - mx = p + d$$

with $|d| < 1/Q$. Setting $q = n - m$, rearranging, and dividing through by q noting that $1 \leq q \leq Q$, we have

$$\left|x - \frac{p}{q}\right| = \frac{d}{q} < \frac{1}{qQ} \leq \frac{1}{q^2}$$

as desired. In particular, for any irrational x and $\varepsilon > 0$ we may choose Q such that $1/Q < \varepsilon$ and hence for some rational p/q have

$$\left|x - \frac{p}{q}\right| < \frac{\varepsilon}{q}$$

showing that $\mathcal{D}(1)$ (and hence its subset $\mathcal{D}(0)$) is vacuous.

□

Theorem 5.10. *(Liouville's Theorem) Every algebraic number is Diophantine. More specifically, if ξ is a* *root of a degree d polynomial $P(x) \in \mathbb{Z}[x]$, then $\xi \in \mathcal{D}(d)$.*

Proof. Let $Z(P)$ denote the zero set of P and $I = [\xi - 1, \xi + 1]$. As I is compact and the map $x \mapsto P'(x)$ is continuous, $\exists M \geq 0$ such that $\forall\, x \in I,\ |P'(x)| \leq M$.

Fix $\frac{p}{q} \in \mathbb{Q} \setminus Z(P)$.

Case 1: $\frac{p}{q} \in I$

If $P(x) = a_d x^d + \cdots + a_0$, we have $P\left(\frac{p}{q}\right) = \frac{1}{q^d}\left|p^d a_d + \cdots + a_0 q^d\right| = \frac{R}{q^d}$. As $a_i, p, q \in \mathbb{Z}$, we have $R \in \mathbb{Z}$ and since $\frac{p}{q}$ is not a root of P, $R \geq 1$ so $\left|P\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^d}$.

Now, by the Mean Value Estimate,

$$\left|\frac{P\left(\frac{p}{q}\right) - P(\xi)}{\xi - \frac{p}{q}}\right| \leq M \implies \frac{1}{q^d} \leq \left|P\left(\frac{p}{q}\right)\right| \leq M\left|\xi - \frac{p}{q}\right|.$$

$$\implies \left|\xi - \frac{p}{q}\right| \geq \frac{1}{Mq^d}.$$

Case 2: $\frac{p}{q} \notin I$

Since $q \in \mathbb{N}_{\geq 1}$, the set

$$((\xi - 1)q, (\xi + 1)q) \cap \mathbb{Z} \neq \emptyset$$

is non-empty so we can take $a \in \mathbb{Z}$ in this set and so $\frac{a}{q} \in I$. Now

$$\frac{p}{q} \notin I \implies \left|\xi - \frac{p}{q}\right| > 1 \geq \left|\xi - \frac{a}{q}\right| \geq \frac{1}{Mq^d}$$

using *case 1.*

We have thus shown that:

$$\left|\xi - \frac{p}{q}\right| \geq \frac{1}{Mq^d}, \quad \forall\, \frac{p}{q} \in \mathbb{Q} \setminus Z(P)$$

As P is a polynomial, $Z(P) \cap \mathbb{Q}$ is a finite set so we can define

$$\delta = \min\left\{ b^d \left|\xi - \frac{a}{b}\right| \;\middle|\; \frac{a}{b} \in Z(P) \cap \mathbb{Q}, \mathrm{hcf}(a, b) = 1 \right\}$$

Setting $0 < \varepsilon < \min\{\delta, \frac{1}{M}\}$ gives the required result. ☐

We can now state the remarkably simple Siegel's linearisation Theorem

Theorem 5.11. *(Siegel, 1942) Let $\lambda = e^{2\pi i\xi}$, $\xi \in \mathbb{R} \setminus \mathbb{Q}$. If ξ is Diophantine of any order, then any germ of a holomorphic function with a fixed point of multiplier λ is locally linearisable.*

The proof will be the focus of the next section. For now, we consider how 'large' the set of such Diophantine ξ is, akin to what we did in the last section.

The notion of 'largness' is captured by the *Lebesgue measure*. We give a brief description in the following and refer the reader to Fremlin (2011) for a comprehensive treatment.

We motivate our definition by the heuristic that we would like an interval $I = (a, b)$ to have measure equal to its length $\ell(I) = b - a$. Furthermore, if a sequence of intervals $\{I_k\}$ covers (that is, contains in its union) a set $S \subseteq \mathbb{R}$, we would expect the measure of S not to exceed the sum of $\ell(I_k)$ across all k. Thus we define the *Lebesgue outer measure* of a set $S \subseteq \mathbb{R}$:

$$\mu^*(S) = \inf \left\{ \sum_{k=0}^{\infty} \ell(I_k) \mid S \subseteq \bigcup_{k=0}^{\infty} I_k \right\}$$

Which possibly takes value ∞ if no such sum converges.

We desire furthermore that the measure behave in a way consistent with the intuitive notion of size, in that we would like a union across a sequence of sets which are mutually disjoint to have measure equal to the sum of its terms. Unfortunately, it happens that it is impossible for all of these properties to hold across all subsets of $\mathbb{R}$ if we insist on assigning a size to every subset. Instead, we define the *Lebesgue measure* only on sets in the *Lebesgue σ-algebra*, which is a collection of subsets of $\mathbb{R}$ that is not all of the powerset of $\mathbb{R}$. There are several ways one might approach this construction; one of them is to define the Lebesgue σ-algebra as the collection of all subsets S that satisfy the *Carathéodory criterion* that

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \setminus A)$$

for all $A \subseteq \mathbb{R}$. Another construction is to take the *Borel σ-algebra* generated by all open sets, and taking its *completion* by including additionally all subsets of $\mathbb{R}$ which have Lebesgue outer measure zero. We write $\mu(S)$ for the Lebesgue measure of S, and for S in the Lebesgue σ-algebra $\mu(S) = \mu^*(S)$.

The Lebesgue measure on $\mathbb{R}/\mathbb{Z}$, as frequently referred to in this report, is obtained by identifying $\mathbb{R}/\mathbb{Z}$ with the interval $[0, 1)$ and taking the measure of subsets accordingly. Some terminology: we say that a property holds *Lebesgue-almost everywhere* on a set X if there is a set N such that the property holds for all $X \setminus N$ and $\mu(N) = 0$.

Lemma 5.12. *Define the set*

$$\mathcal{D}(2+) = \bigcap_{\kappa>2} \mathcal{D}(\kappa)$$

which consists of all integers that are Diophantine of every order $\kappa > 2$. Then $\mathcal{D}(2+)$ has full measure on the circle $\mathbb{R}/\mathbb{Z}$ (that is, its compliment has measure zero).

Proof. We construct an open covering of $\mathcal{D}(2+)^c$ which has measure converging to zero. We have

$$\mathcal{D}(\kappa)^c = \left\{ \xi \in [0,1) \;\middle|\; \forall \varepsilon > 0,\, \exists\, \frac{p}{q} \in \mathbb{Q} \cap [0,1) \ s.t. \ \left| \xi - \frac{p}{q} \right| \leq \frac{\varepsilon}{q^\kappa} \right\}$$

Define the set

$$U(\kappa, \varepsilon) = \left\{ \xi \in [0,1) \;\middle|\; \exists\, \frac{p}{q} \in \mathbb{Q} \cap [0,1) \ s.t. \ \left| \xi - \frac{p}{q} \right| < \frac{\varepsilon}{q^\kappa} \right\}$$

which is a union of open intervals so is an open set. We also have

$$\mathcal{D}(\kappa)^c = \bigcap_{\varepsilon>0} U(\kappa, \varepsilon)$$

Fix $q \in \mathbb{Z}_{\geq 1}$. Then we have q possible choices for p so that $\frac{p}{q} \in [0,1)$. Moreover, if $\xi \in U(\kappa,\varepsilon)$, then $\xi \in \left(\frac{p}{q} - \frac{\varepsilon}{q^\kappa}, \frac{p}{q} + \frac{\varepsilon}{q^\kappa}\right)$ and so we deduce

$$U(\kappa, \varepsilon) \subseteq \bigcup_{q>1} \left\{ \bigcup_{p=0}^{q-1} \left(\frac{p}{q} - \frac{\varepsilon}{q^\kappa}, \frac{p}{q} + \frac{\varepsilon}{q^\kappa} \right) \right\}$$

$\implies \left\{ \bigcup_{p=0}^{q-1} \left(\frac{p}{q} - \frac{\varepsilon}{q^\kappa}, \frac{p}{q} + \frac{\varepsilon}{q^\kappa} \right) \;\middle|\; q \in \mathbb{Z}_{\geq 1} \right\}$ is an open cover of $U(\kappa, \varepsilon)$.

Hence, $U(\kappa,\varepsilon)$ has measure $\leq \sum_{q=1}^{\infty} \frac{2\varepsilon}{q^{\kappa-1}}$. If $\kappa > 2$ this sum converges and so the upper bound $\longrightarrow 0$ as $\varepsilon \longrightarrow 0$. As $\mathcal{D}(\kappa)^c \subseteq U(\kappa,\varepsilon)\ \forall \varepsilon > 0$, it follows that $\mathcal{D}(\kappa)^c$ has measure zero and so

$$\mathcal{D}(2+)^c = \bigcup_{\kappa>2} \mathcal{D}(\kappa)^c$$

has measure zero. ☐

Corollary 5.13. *For every ξ outside of a set of measure zero, we can conclude that every holomorphic germ* *with a fixed point of multiplier $e^{2\pi i\xi}$ is locally linearisable.*

Proof. Immediate from lemma (5.12) and theorem (5.11). □

This reveals a startling dichotomy between a *general* property in the topological sense and a property that occurs *almost everywhere* in the vernacular of measure theory. According to Milnor, 2006, this contrast is *'not uncommon in dynamics'*.

We will present a proof of Siegel's 1942 linearisation theorem (5.11) in a latter section. But first we present a proof of a weaker version of the theorem, due to Yoccoz and following the presentation in Milnor. In some cases this version of the proof also gives us a way to estimate the 'size' of a Siegel disc in the sense that its *conformal radius* is the limit of a sequence that can be computed recursively.

5.4 Quadratic Siegel Discs

In this section we consider maps of the form $f_\lambda(z) = \lambda z + z^2$. The main result is that Siegel discs actually exist: indeed for Lebesgue-almost all values of λ where $|\lambda| = 1$, f_λ has a Siegel disc about the origin.

5.4.1 The Conformal Radius Function

But first, some complex analysis.

Definition 5.14 (conformal radius)**.** Let U a proper subset of $\mathbb{C}$ be a simply connected domain. By the Riemann mapping theorem, for each $z_0 \in U$ there is a unique conformal isomorphism $\phi : U \to \mathbb{D}$ with $\phi(z_0) = 0$ and $\phi'(z_0) > 0$. In terms of this map the *conformal radius of U from z_0* is defined as

$$z_0 U = 1/\phi'(z_0)$$

Intuitively, the conformal radius captures the *size* of a simply connected domain in a sense that is invariant under conformal isomorphisms. This may be easier seen from the equivalent definition of $z_0 U$ as the unique $r \geq 0$ such that there is a conformal isomorphism $\psi : U \to \mathbb{D}_r$ to the disc with radius r fixing the origin and such that $\psi'(0) = 1$.

We are concerned with Siegel discs, which are connected components of the Fatou set on which f is conformally conjugate to a rotation of the unit disc. It is an immediate consequence of the definition that a Siegel disc must be conformally isomorphic to $\mathbb{D}$. Therefore the following definition makes sense:

Definition 5.15 (conformal radius function)**.** For each $\lambda \in \mathbb{C}$, define $\sigma(\lambda)$ to be the conformal radius from 0 of the maximal linearising neighbourhood of f_λ about the origin, taking $\sigma(\lambda) = 0$ if no such neighbourhood can exist.

More explicitly, $\sigma(\lambda)$ is the maximal r such that there exists a univalent map $\psi_\lambda : \mathbb{D}_\sigma \to \mathbb{C}$ such that

- $\psi_\lambda(0) = 0$ and $\psi'_\lambda(0) = 1$, and
- $f_\lambda(\psi_\lambda(w)) = \psi_\lambda(\lambda w)$, that is, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f_\lambda} & \mathbb{C} \\ {\scriptstyle\psi_\lambda}\uparrow & & \uparrow{\scriptstyle\psi_\lambda} \\ \mathbb{D}_\sigma & \xrightarrow[w\mapsto\lambda w]{} & \mathbb{D}_\sigma \end{array}$$

Immediately we note the following:

- $\sigma(\lambda)$ is positive for $0 < |\lambda| < 1$ from Koenigs linearisation (Theorem 2.4)
- $\sigma(\lambda) = 0$ when $\lambda = 0$ as a consequence of Böttcher's theorem on superattracting fixed points (3.1).

In particular, σ is nonconstant on $\overline{\mathbb{D}}$. In addition, we establish the following properties of the conformal radius function:

Lemma 5.16. *The conformal radius function σ is bounded and upper semi-continuous on $\overline{\mathbb{D}}$. Furthermore, there exists holomorphic $\eta : \mathbb{D} \to \mathbb{C}$ such that $|\eta(\lambda)| = \sigma(\lambda)$ for all $|\lambda| \in \mathbb{D}$.*

Recalling that a real-valued function $f : \mathbb{C} \to \mathbb{R}$ is *upper semi-continuous* if the superlevel set

$$L_c^+(f) = \{z \in \mathbb{C} \mid c \leq f(z)\}$$

is closed for every $c \in \mathbb{R}$.

We postpone the proof of Lemma 5.16. For now assuming its conclusion, we see first how therefrom to arrive at the existence of quadratic Siegel discs.

5.4.2 Quadratic Siegel Discs Exist

We state now the main result of this section, which is a weaker version of Siegel's linearisation theorem 5.11:

Theorem 5.17. *For Lebesgue almost-every $\xi \in \mathbb{R}/\mathbb{Z}$, $f_\lambda : z \mapsto z^2 + e^{2\pi i \xi} z$ has a Siegel disc about the origin.*

We shall establish this result by way of the following lemma, the proof of which we do not present here.

Lemma 5.18 (F. and M. Riesz, 1916). *Let $\eta : \mathbb{D} \to \mathbb{C}$ be bounded and holomorphic. If for some constant $c \in \mathbb{C}$ the set of ξ such that*

$$\lim_{r \nearrow 1} \eta(re^{2\pi i \xi}) = c$$

has positive Lebesgue measure, then η is constant.

Proof of Theorem 5.17. Fix some $\lambda_0 = e^{2\pi i \xi}$. If f_{λ_0} is not linearisable near the origin (i.e. if f_{λ_0} has a Cremer point or a parabolic point there), then $\sigma(\lambda_0) = 0$. This happens if and only if

$$\lim_{\lambda \to \lambda_0} \eta(\lambda) = 0$$

with $\lambda \in \mathbb{D}$: see this by noting that we always have

$$\lim_{\lambda \to \lambda_0} |\eta(\lambda)| = \lim_{\lambda \to \lambda_0} \sigma(\lambda) \leq \sigma(\lambda_0)$$

in which left-hand-side limit exists because η is bounded and holomorphic, the inequality follows from upper semi-continuity of σ, and the inequality is in fact an equality when $\sigma(\lambda_0) = 0$ since σ is nonnegative.

At this point we invoke Lemma 5.18, with $c = 0$ and taking the limit along $\lambda = re^{2\pi i \xi}$ for $r < 1$. Since η is nonconstant, we have then that the values for $\xi \in \mathbb{R}/\mathbb{Z}$ such that f_λ has no local linearisation must have Lebesgue measure zero. □

To complete the argument, it remains to prove Lemma 5.16.

Proof of Lemma 5.16. We treat each part of the statement in turn.

- *σ is bounded.* Indeed $\sigma(\lambda) \leq 2$ for all $\lambda \in \overline{\mathbb{D}}$: see that for $|z| > 2$, $|z + \lambda| > 1$ on $\overline{\mathbb{D}}$, and thus

$$|f_\lambda(z)| = |z(z + \lambda)| > |z|$$

  bounds $\{|f_\lambda^n(z)|\}$ below by a diverging geometric series. Therefore the iterated images of such a z diverges to infinity and cannot lie within a neighbourhood of the origin in which f_λ is conjugated to a rotation. Therefore any $\psi_\lambda : \mathbb{D}_\sigma \to \mathbb{C}$ as in Definition 5.15 maps $\mathbb{D}_\sigma$ into $\mathbb{D}_2 \subseteq \mathbb{C}$; further if $\psi'_\lambda(0) = 1$ then by the

Cauchy derivative estimate

$$1 = |\psi'_\lambda(0)| \leq 2/\sigma$$

so $\sigma(\lambda) \leq 2$, as desired.

- σ *is upper semi-continuous.* We show that the superlevel set $L^+_{\sigma_0}(\sigma)$ is closed for each σ_0 by showing that it is sequentially closed.

  Take any sequence $\{\lambda_k\}$ in $L^+_{\sigma_0}(\sigma)$ such that $\lambda_k \longrightarrow \lambda$ as $k \longrightarrow \infty$. To each λ_k there corresponds a map $\psi_{\lambda_k} : \mathbb{D}_{\sigma_0} \to \mathbb{D}_2$ which conjugates f_{λ_k} to a rotation on $\mathbb{D}_{\sigma_0}$ (indeed by hypothesis there is such a map which conjugates f_{λ_k} to a rotation on $\mathbb{D}_{\sigma(\lambda_k)}$ where $\sigma_0 \leq \sigma(\lambda_k)$; we obtain ψ_{λ_k} by restricting the domain). The family $\{\psi_{\lambda_k}\}$ is normal since it is a family of maps between open discs (Theorem 6.11 in the Appendix), hence we may extract a convergent subsequence with limit ψ_λ.

  Now ψ_λ is univalent since it is a local uniform limit of univalent functions (Theorem 6.7), and it remains only to verify that ψ_λ indeed conjugates f_λ to a rotation of $\mathbb{D}_\sigma$. By construction

$$\psi_\lambda(0) = 0 \quad \text{and} \quad \psi'_\lambda(0) = 1$$

  where the convergence of ψ'_λ follows from the Weierstrass uniform convergence theorem. Now on $\mathbb{D}_{\sigma_0}$:

$$\begin{aligned}
\psi_\lambda(\lambda w) &= (\lim_{k\to\infty} \psi_{\lambda_k})(\lim_{l\to\infty} \lambda_l w) \\
&= \lim_{l\to\infty} (\lim_{k\to\infty} \psi_{\lambda_k})(\lambda_l w) && \text{(by continuity of } \psi_\lambda) \\
&= \lim_{k\to\infty} \psi_{\lambda_k}(\lambda_k w) && \text{(convergence of a subsequence)} \\
&= \lim_{k\to\infty} f_{\lambda_k}(\psi_{\lambda_k}(w)) \\
&= f_\lambda(\psi_\lambda(w)) && (f_{\lambda_k} \longrightarrow f_\lambda \text{ locally uniformly})
\end{aligned}$$

  Therefore ψ_λ conjugates f_λ to a rotation, thus $\lambda \in L^+_{\sigma_0}$ as desired.

- σ *coincides with the absolute value of some holomorphic* $\eta : \mathbb{D} \to \mathbb{C}$ *on the unit disc* $\mathbb{D}$. We define for $0 < |\lambda| < 1$

$$\eta(\lambda) = \phi(-\lambda/2)$$

  where ϕ is the Koenigs co-ordinate as in Theorem 2.7 and is holomorphic throughout the punctured disc. Further by Theorem 2.8 we may extend ψ_λ onto $\overline{\mathbb{D}}_\sigma$, and $\psi_\lambda(\partial\mathbb{D}_\sigma)$ contains a critical point. But

$z^\# = -\lambda/2$ is the only critical point of f_λ, so there is $z \in \psi_\lambda^{-1}(z^\#)$ satisfying

$$\sigma(\lambda) = |z| = |\phi(\psi_\lambda(z))| = \left|\phi(z^\#)\right| = |\eta(\lambda)|$$

for every λ in the punctured disc. Finally we observe that

$$\lim_{\lambda \to 0} |\eta(\lambda)| = \lim_{\lambda \to 0} \sigma(\lambda) \leq \sigma(0) = 0$$

where the inequality follows from upper semi-continuity, therefore $\eta(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow 0$; setting $\eta(0) = 0$ we thereby extend η to a holomorphic function on the unit disc with the desired property.

Together these establish the conclusion of Lemma 5.16, and in turn complete the proof of Theorem 5.17 □

5.4.3 Computing the Conformal Radius

Recall that we constructed the Koenigs co-ordinate in Theorem 2.7 by the locally uniform limit

$$\phi_\lambda(z) = \lim_{n \to \infty} \frac{f_\lambda^{\circ n}(z)}{\lambda^n}$$

Since $\sigma(\lambda) = |\eta(\lambda)| = \left|\phi_\lambda(z^\#)\right|$, this gives us a way to compute the conformal radius as the limit of the sequence $\{\eta_k\}$ whose terms are $\eta_k = f_\lambda^n(z^\#)/\lambda^k$. The η_k are given by the recurrence relation

$$\begin{aligned} \eta_{k+1} &= \frac{f_\lambda^{\circ(k+1)}(z^\#)}{\lambda^{k+1}} \\ &= \frac{\left(f_\lambda^{\circ k}(z^\#)\right)^2 + \lambda f_\lambda^{\circ k}(z^\#)}{\lambda^{k+1}} \\ &= \lambda^{k-1}\left(\frac{f_\lambda^{\circ k}(z^\#)}{\lambda^k}\right)^2 + \frac{f_\lambda^{\circ k}(z^\#)}{\lambda^k} = \lambda^{k-1}\eta_k^2 + \eta_k \end{aligned}$$

with $\eta_0 = z^\# = -\lambda/2$. The next few terms in this sequence are

$$\begin{aligned} \eta_1 &= \lambda^{-1}\left(-\frac{\lambda}{2}\right)^2 + \left(-\frac{\lambda}{2}\right) = \frac{\lambda}{4} \\ \eta_2 &= \lambda^0\left(\frac{\lambda}{4}\right)^2 + \frac{\lambda}{4} = \frac{\lambda}{4} + \frac{\lambda^2}{16} \\ \eta_3 &= \lambda^1\left(\frac{\lambda}{4} + \frac{\lambda^2}{16}\right)^2 + \frac{\lambda}{4} + \frac{\lambda^2}{16} = \frac{\lambda}{4} + \frac{\lambda^2}{16} + \frac{\lambda^3}{16} + \frac{\lambda^4}{32} + \frac{\lambda^5}{256} \end{aligned}$$

In particular note that $\eta_{k+1} - \eta_k$ has leading term of the order of λ^k. Therefore η_{k+1} agrees with η_k up to the order of λ^{k-1}, and we may continue this procedure to compute the coefficients of the power seires expansion of $\eta(\lambda)$, the first few terms of which are, after some more computation:

$$\eta(\lambda) = -\frac{\lambda}{4} + \frac{\lambda^2}{16} + \frac{\lambda^3}{16} + \frac{\lambda^4}{32} + \frac{9\lambda^5}{256} + \frac{\lambda^6}{256} + \frac{7\lambda^7}{256} + O(\lambda^8)$$

5.5 Proof of Siegel's Linearisation Theorem

We present a proof different from Siegel's more involved original, following Gamelin (2013).

Given f fixing the origin with multiplier λ, we would like to construct a function $\psi : \mathbb{D}_r \to \mathbb{C}$ univalent on some neighbourhood of the origin satisfying Schröder's equation:

$$f(\psi(z)) = \psi(\lambda z)$$

Without loss of generality we may require $\psi(0) = 0$ and $\psi'(0) = 1$; then up to first order, we can write

$$\psi(z) = z + \sum_{j=2}^{\infty} a_j z^j = z + \Psi(z) \quad \text{and} \quad f(z) = \lambda z + \sum_{j=2}^{\infty} b_j z^j = \lambda z + F(z)$$

Substituting into Schröder's equation

$$\lambda(z + \Psi(z)) + F(z + \Psi(z)) = \lambda z + \Psi(\lambda z) \tag{23}$$

Heuristically, we expect $\Psi(z)$ to be small and imagine $F(z + \Psi(z)) \approx F(z)$. Then, replacing one term with the other and rearranging, of the previous equation (23) remains

$$F(z) = \Psi(\lambda z) - \lambda \Psi(z) \tag{24}$$

and we expect that a solution to (24) to approximately solve the simpler equation (23). By inspecting the coefficients, (24) is satisfied by

$$\Psi_0(z) = \sum_{j=2}^{\infty} \frac{b_j}{\lambda^j - \lambda} z^j$$

Tentatively we define

$$\psi_0(z) = z + \Psi_0(z) \tag{25}$$

Which we can make into an injection by restricting to a smaller domain. The strategy is to iterate this process in an attempt to progressively improve the approximate solutions. The content of this proof lies in the estimations that ensure the convergence of the iterates on some neighbourhood of the origin to a univalent function which satisfies Schröder's equation.

The following lemma captures a step in this iterative process:

Lemma 5.19. *Suppose* λ *is such that there exists* C, κ *such that for all* $j \in \mathbb{N}$

$$\left|\lambda^j - 1\right| \leq C j^\kappa$$

and let

$$f(z) = \lambda z + \sum_{j=2}^{\infty} b_j z^j = \lambda z + F(z)$$

defined in some neighbourhood of the origin which has multiplier λ*. Suppose furthermore that* δ, η *satisfy the following:*

- $0 < \eta < \dfrac{1}{5}$ *and* $\dfrac{\eta}{1-\eta} < \dfrac{\kappa}{2^{\kappa+2}}$
- $c\delta < \eta^{\kappa+2}$
- $|F'(z)| < \delta$ *for* $z \in \mathbb{D}_r$

in which we define the constant $c = \max(1, \kappa! C)$*. Then there exist holomorphic* $\psi, g : \mathbb{D}_{r(1-5\eta)} \to C$ *where*

$$g(z) = \lambda z + G(z) \quad \textit{and} \quad \psi(z) = z + \Psi(z)$$

such that

- $g(z) = (\psi^{-1} \circ f \circ \psi)(z)$
- $|\Psi'(z)| \leq \eta$
- $|G'(z)| \leq \dfrac{c\delta^2}{\eta^{\kappa+2} 2^{\kappa+2}}$

for $z \in \mathbb{D}_{r(1-5\eta)}$*.*

The proof of Lemma (5.19) is somewhat involved, and we shall postpone its presentation briefly. Let us first see how we can use this result to construct the promised iteration process that will lead to a solution to Schröder's equation and yield the desired local linearisation of f.

Proof of Theorem 5.11 assuming Lemma 5.19. Let $\lambda = e^{2\pi i \xi}$ where ξ is Diophantine of some order $\kappa + 1$, and let $f(z)$ fix the origin with multiplier λ. We write

$$f_0(z) = f(z) = \lambda z + F_0(z)$$

And by Lemma5.9 we have $\left|\lambda^j - 1\right| \leq Cj^\kappa \leq c\frac{j^\kappa}{\kappa!}$ for some constants κ, c, and for all $j \in \mathbb{N}$. Now that c and κ are fixed, we fix η_0, choosing it to be small enough such that it satisfies

$$0 < \eta_0 < \frac{1}{5} \quad \text{and} \quad \frac{\eta_0}{1-\eta_0} < \frac{\kappa}{2^{\kappa+2}}$$

We then fix $\delta_0 > 0$ sufficiently small so that $c\delta_0 < \eta_0^{\kappa+2}$, then fix $r_0 > 0$ sufficiently small so that f is defined on $\mathbb{D}_{r_0}$ and $|F_0'(z)| \leq \delta_0$ on $\mathbb{D}_{r_0}$. Then by construction, the hypotheses on f, η, r, δ in Lemma 5.19 are satisfied by $f_0, \eta_0, r_0, \delta_0$.

Suppose now that for some $n \in \mathbb{N}$ those hypotheses are also satisfied by $f_n, \eta_n, r_n, \delta_n$. We define

- $\eta_{n+1} = \frac{1}{2}\eta_n$
- $r_{n+1} = r_n(1 - 5\eta_n)$
- $\delta_{n+1} = \frac{c\delta_n^2}{\eta_{n+1}^{\kappa+2}}$

Evidently $0 < \eta_{n+1} < \frac{1}{5}$ and $\frac{\eta_{n+1}}{1-\eta_{n+1}} < \frac{\kappa}{2^{\kappa+2}}$. We also note that

$$\begin{aligned}
c\delta_{n+1} &= \frac{c^2\delta_n^2}{\eta_n^{\kappa+2}2^{\kappa+2}} && \text{(definition of } \delta_{n+1} \text{ and } \eta_{n+1}) \\
&< \frac{(\eta_n^{\kappa+2})^2}{\eta_n^{\kappa+2}} && \text{(inductive hypothesis } c\delta_n \leq \eta_n^{\kappa+2}) \\
&= \frac{\eta_n^{\kappa+2}}{2^{\kappa+2}} = \eta_{n+1}^{\kappa+2}
\end{aligned}$$

At this point we invoke Lemma 5.19 to obtain $\psi_n, f_{n+1} : \mathbb{D}_{r_{n+1}} \to \mathbb{C}$ such that throughout $\mathbb{D}_{r_{n+1}}$, writing $f_{n+1}(z) = \lambda z + F_{n+1}(z)$, we have

- $\psi_n \circ f_{n+1} = f_n \circ \psi_n$

- $\left|F'_{n+1}(z)\right| < \dfrac{c\delta_n^2 r}{\eta_n^{\kappa+2} 2^{\kappa+2}} = \delta_{n+1}$

Thus the hypotheses on f, η, r, δ in Lemma 5.19 are satisfied by $f_{n+1}, \eta_{n+1}, r_{n+1}, \delta_{n+1}$.

Therefore, inductively, we have sequences $\{\psi_n\}, \{f_n\}, \{\eta_n\}, \{r_n\}, \{\delta_n\}$

$$\begin{array}{ccc}
\cdot & \xrightarrow{\;f_0 = f\;} & \cdot \\
\psi_0 \uparrow & & \downarrow \psi_0^{-1} \\
\cdot & \xrightarrow{\;f_1 = \psi_0^{-1} f \psi_0\;} & \cdot \\
\psi_1 \uparrow & & \downarrow \psi_1^{-1} \\
\cdot & \xrightarrow{\;f_2 = \psi_1^{-1}\psi_0^{-1} f \psi_0 \psi_1\;} & \cdot \\
\psi_2 \uparrow & & \downarrow \psi_2^{-1} \\
\cdot & \xrightarrow{\;f_3 = \psi_2^{-1}\psi_1^{-1}\psi_0^{-1} f \psi_0 \psi_1 \psi_2\;} & \cdot \\
\psi_3 \uparrow & & \downarrow \psi_3^{-1} \\
\vdots & & \vdots
\end{array}$$

To finally extract the desired conjugation, we must be careful to verify that there is a neighbourhood about the origin on which all the f_n are defined and converge. First, see that $\sum_{n\geq 0} \eta_n = \sum_{n\geq 0} \eta_0 2^{-n}$ converges as a geometric series, so by a result on infinite series (Theorem 6.14) the product

$$r_\infty = r_0 \prod_{n\geq 0} (1 - 5\eta_n)$$

converges to a positive number. We see from this that there is indeed a nonempty disc $\mathbb{D}_{r_\infty}$ contained within $\mathbb{D}_{r_n}$ for every n, and every f_n is defined on $\mathbb{D}_{r_\infty}$. Writing $\bar{\psi}_n = \psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_0$, we have the following bound:

$$\begin{aligned}
\sup_{z\in\mathbb{D}_{r_\infty}} \left|\bar{\psi}'_n(z)\right| &\leq \prod_{k=0}^{n} |\psi'_k(z)| && \text{(chain rule)} \\
&\leq \prod_{k=0}^{n} \sup_{z\in\mathbb{D}_{r_\infty}} |1 + \eta_k| && (|\Psi'_k| \leq \eta_k \text{ from Lemma 5.19}) \\
&\leq \prod_{k\geq 0} |1 + \eta_k| \leq M && \text{in which } M \text{ is a constant.}
\end{aligned}$$

The final product converges since $\sum_{k\geq 0} \eta_k$ does (by a result on infinite series, see Theorem 6.13 in the Appendix.) By possibly replacing M with a greater bound we may assume $M \geq 1$. Then by the Cauchy derivative estimate we have a neighbourhood $\mathbb{D}_{r_\infty/M}$ of the origin which is mapped by $\bar{\psi}_n$ into $\mathbb{D}_{r_\infty}$, subsequently the mean value inequality on $\mathbb{D}_{r_\infty/M}$ gives

$$\begin{aligned}\left|\bar{\psi}_{n+1}(z)-\bar{\psi}_n(z)\right| &= \left|\psi_{n+1}(\bar{\psi}_n(z))-\bar{\psi}_n(z)\right| \\ &= \left|\Psi_{n+1}(\bar{\psi}_n(z))\right| && \text{(by } \psi_{n+1}(z)=z+\Psi_{n+1}(z)\text{)} \\ &\le \eta_{n+1} && (\bar{\psi}_n(z)\in\mathbb{D}_{r_\infty};\text{ Lemma 5.19)}\end{aligned}$$

From this and the convergence of $\sum_{n\geq 0}\eta_n$ we see that $\{\bar{\psi}_n\}$ is Cauchy in the uniform norm and thus the sequence of functions converges uniformly on $\mathbb{D}_{r_\infty/M}$, and the limit $\bar{\psi}$ is a holomorphic bijection since each of the $\bar{\psi}_n$ are.

Furthermore by the bound

$$|F_n'(z)| < \delta_n \longrightarrow 0$$

we have on $\mathbb{D}_{r_\infty/M}\subseteq\mathbb{D}_{r_\infty}$ that F_n converges uniformly to zero and thus f_n converges uniformly to $z\mapsto\lambda z$. Therefore we finally have

$$(\bar{\psi}^{-1}\circ f\circ\bar{\psi})(z)=\lambda z$$

for all z in some neighbourhood of the origin, demonstrating that f is locally linearisable. □

To complete the proof of Siegel's linearisation theorem, it remains to prove Lemma 5.19, which we do in the following.

Proof of Lemma 5.19. We define

$$\Psi(z)=\sum_{k=2}^{\infty}\frac{b_j}{\lambda^j-\lambda}z^k$$

- *Bound on* $|\Psi|$. See first that for $z\in\mathbb{D}_r$ we have $|F(z)|\leq\delta r$ from $|F'(z)|\leq\delta$ and the mean value inequality; therefore

$$|b_j| = \left|\frac{f^{(j)}(0)}{j!}\right| \leq \frac{1}{j!}\cdot\delta r j! r^{-j} = \frac{\delta}{jr^{j-1}} \tag{26}$$

  on $\mathbb{D}_r$, in which the inequality follows from the Cauchy derivative estimate for $\left|F^{(n)}(z)\right|$.

Then throughout the slightly smaller disc $\mathbb{D}_{r(1-\eta)}$ we have

$$\begin{aligned}
|\Psi(z)| &= \left|\sum_{j=2}^{\infty} \frac{b_j}{\lambda^j - \lambda} z^j\right| \\
&< \sum_{j=2}^{\infty} \frac{|b_j|}{|\lambda(\lambda^{j-1}-1)|} \left(r(1-\eta)\right)^j && \text{(since } |z| < r(1-\eta)\text{)} \\
&\le \sum_{j=2}^{\infty} \frac{\delta}{j r^{j-1}} \cdot c \frac{(j-1)^\kappa}{\kappa!} r^j (1-\eta)^j && \text{(by (26) and hypothesis on } \lambda\text{)} \\
&\le \frac{c\delta r}{\kappa!} \sum_{j=2}^{\infty} j^{\kappa-1} (1-\eta)^j && \text{(rearranging, } (j-1)^k \le j^k\text{)} \\
&\le \frac{c\delta r}{\kappa} \sum_{j=2}^{\infty} \frac{(j+\kappa-1)!}{j!(\kappa-1)!} (1-\eta)^j \\
&= \frac{c\delta r}{\kappa} \sum_{j=2}^{\infty} \binom{j+\kappa-1}{j} (1-\eta)^j \le \frac{c\delta r}{\kappa \eta^\kappa}
\end{aligned}$$

in which the final inequality follows from considering the binomial expansion of $\eta^{-\kappa}$ (Theorem 6.12)

- *Bound on* $|\Psi'|$. This is a very similar calculation to the previous one:

$$\begin{aligned}
|\Psi'(z)| &< \sum_{j=2}^{\infty} \frac{j\,|b_j|}{|\lambda(\lambda^{j-1}-1)|} \left(r(1-\eta)\right)^{j-1} \\
&\le \frac{c\delta}{\kappa!} \sum_{j=2}^{\infty} (j-1)^\kappa r^{j-1} \\
&\le \frac{c}{\delta} \sum_{j=1}^{\infty} \binom{j+\kappa}{j} (1-\eta)^j && \text{(relabelling; definition of binomial coefficients)} \\
&\le \frac{c\delta}{\eta^{\kappa+1}} && \text{(Theorem 6.12)} \\
&< \eta && \text{(by hypothesis } c\delta < \eta^{\kappa+2}\text{)} \qquad (27)
\end{aligned}$$

- $g = \psi^{-1} \circ f \circ \psi$ *is well-defined on* $\mathbb{D}_{r(1-4\eta)}$. This claim is a combination of several parts, and we verify each of them in turn:
  - ψ *maps* $\mathbb{D}_{r(1-4\eta)}$ *into* $\mathbb{D}_{r(1-3\eta)}$. For $z \in \mathbb{D}_{r(1-4\eta)}$ we have

$$|\psi(z)| = |z + \Psi(z)| \le r(1-4\eta) + r(1-4\eta)\cdot\eta < r(1-3\eta)$$

in which the first inequality follows from the triangle inequality, the mean value inequality, and the estimate (27) of $|\Psi'(z)| < \eta$.

– *f maps $\mathbb{D}_{r(1-3\eta)}$ into $\mathbb{D}_{r(1-2\eta)}$.* Similarly since $|F'(z)| < \delta < \frac{\eta^{\kappa+2}}{c} \leq \eta$ on $\mathbb{D}_r$, we have

$$|f(z)| = |\lambda z + F(z)| \leq r(1-3\eta) + \delta \cdot r(1-3\eta) < r(1-2\eta)$$

– *$\psi^{-1} : \mathbb{D}_{r(1-2\eta)} \to \mathbb{D}_{r(1-\eta)}$ is well-defined.* That is, we show that for every $y \in \mathbb{D}_{r(1-2\eta)}$, there is exactly one $z \in \mathbb{D}_{r(1-\eta)}$ such that $y = \psi(z)$. By yet another similar estimate we have

$$|\psi(z)| = |z + \Psi(z)| > r(1-\eta) - \eta \cdot r(1-\eta) \geq r(1-2\eta)$$

Note first that $\psi(0) = 0$ by construction. Then by the ML-inequality we estimate

$$\begin{aligned} \left| \frac{1}{2\pi i} \oint_\gamma \frac{\psi'(z)}{\psi(z)} dz \right| &\leq \frac{1}{2\pi} \cdot 2\pi r(1-\eta) \cdot \frac{1+\eta}{r(1-2\eta)} \qquad \text{(since } |\psi'(z)| \leq 1 + |\Psi'(z)| \leq 1+\eta) \\ &= \frac{1-\eta^2}{1-2\eta} < 2 \end{aligned}$$

where the integral is along the boundary $\gamma = \partial\mathbb{D}_{r(1-\eta)}$. Then by the argument principle, ψ has at most — and hence exactly — one root in $\mathbb{D}_{r(1-\eta)}$. Furthermore, for any $y \in \mathbb{D}_{r(1-2\eta)}$, for z on the boundary of $\mathbb{D}_{r(1-\eta)}$ we have

$$|(\psi(z) - y) - \psi(z)| = |y| < r(1-2\eta) \leq |\psi(z)|$$

and so by the Rouché's theorem applied to $\psi(z) - y$ and $\psi(z)$ on $\mathbb{D}_{r(1-2\eta)}$ we obtain the desired conclusion that $\psi(z) - y$ has exactly one root in $\mathbb{D}_{r(1-\eta)}$.

Together we have that the maps in the following diagram are well-defined, and that the diagram commutes:

$$\begin{array}{ccc} \mathbb{D}_{r(1-3\eta)} & \xrightarrow{\quad f \quad} & \mathbb{D}_{r(1-2\eta)} \\ \psi \uparrow & & \downarrow \psi^{-1} \\ \mathbb{D}_{r(1-4\eta)} & \xrightarrow[g=\psi^{-1}\circ f\circ\psi]{} & \mathbb{D}_{r(1-\eta)} \end{array}$$

thus we may define $g(z) = \psi^{-1} \circ f \circ \psi(z) = \lambda z + G(z)$ on $\mathbb{D}_{r(1-4\eta)}$ (that the first term of the series expansion of g at the origin is λz follows from the chain rule.)

- *bound on* $|G|$. Substituting $g(z) = \lambda z + G(z)$ into $\psi \circ g = f \circ \psi$ we obtain, after some computation:

$$G(z) = \lambda\Psi(z) - \Psi(\lambda z + G(z)) + F(z + \Psi(z)) = \Psi(\lambda z) - F(z) - \Psi(\lambda z + G(z)) + F(z + \Psi(z))$$

  in which the second equality holds noting that

$$\Psi(\lambda z) - \lambda\Psi(z) = F(z)$$

  from the definition of Ψ in terms of F.

  Writing $\Delta = \mathbb{D}_{r(1-4\eta)}$ and $M = \sup_{z\in\Delta} |G(z)|$, see by the triangle and mean value inequalities together with the bounds on $|\Psi|$, $|\Psi'|$, and $|F'|$ that

$$\begin{aligned} M &\leq \sup_{z\in\Delta} |\Psi(\lambda z) - \Psi(\lambda z + G(z))| + \sup_{z\in\Delta} |F(z + \Psi(z)) - F(z)| \\ &\leq \sup_{z\in\Delta} |\Psi'(z)| \cdot M + \sup_{z\in\Delta} |F'(z)| \cdot \sup_{z\in\Delta} |\Psi(z)| \\ &< \eta \cdot M + \delta \cdot \frac{c\delta r}{\kappa\eta^\kappa} \end{aligned}$$

  Consequently

$$M \leq \frac{c\delta^2 r}{\kappa\eta^\kappa(1-\eta)}$$

- *bound on* $|G'|$. For any $z \in \mathbb{D}_{r(1-5\eta)}$, consider the disc around z of radius $r\eta$, which is contained entirely within $\mathbb{D}_{r(1-4\eta)}$. Then the Cauchy derivative estimate gives

$$\begin{aligned} |G'(z)| &\leq \frac{M}{r\eta} \\ &\leq \frac{c\delta^2}{\kappa\eta^{\kappa+1}(1-\eta)} \\ &\leq \frac{c\delta^2}{\kappa\eta^{\kappa+2}2^{\kappa+2}} \end{aligned}$$

  for all $z \in \mathbb{D}_{r(1-5\eta)}$, as desired. The final inequality follows from the hypothesis $\dfrac{\eta}{1-\eta} < \dfrac{\kappa}{2^{\kappa+2}}$.

$\square$

5.6 Return Times

If we have a multiplier $\lambda = e^{2\pi i\xi}$ for ξ an irrational rotation, a natural question is, for a given initial point, how close and how often does the orbit return to a neighbourhood of the initial point?

Such questions are common in dynamics and in this case, precise answers can be given with results from classical number theory. The results will have implications to both celestial mechanics and so-called *small divisors problems.*

Let $S^1 \subset \mathbb{C}$ be the unit circle. We focus on the map $z \mapsto \lambda z$ for $z \in S^1$. Now without loss of generality, we can assume $z = 1$ by rotating the unit circle if necessary. We thus study the orbit

$$1 \longmapsto \lambda \longmapsto \lambda^2 \longmapsto \lambda^3 \longmapsto \dots$$

As ξ is irrational, 1 is not a periodic point so $\{1, \lambda, \lambda^2, \dots\}$ are all distinct.

Definition 5.20. The expression

$$\xi = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

for $a_n \in \mathbb{N}$ is the *continued fraction of* ξ. The n^{th} truncated fraction

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_{n-1}}}}$$

is called the n^{th} *convergent to* ξ.

Lemma 5.21. *If ξ has the above continued fraction decomposition, then*

1. $p_0 = 1, p_1 = 0, q_0 = 0, q_1 = 1$ *and*

$$p_{n+1} = a_n p_n + p_{n-1}$$

$$q_{n+1} = a_n q_n + q_{n-1}$$

2. $\frac{p_n}{q_n} < \xi$ *if n is odd and* $\frac{p_n}{q_n} > \xi$ *if n is even*

3. $p_n q_{n+1} - p_{n+1} q_n = (-1)^n$

4.

$$0 = \frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_5}{q_5} < \cdots < \xi < \cdots < \frac{p_6}{q_6} < \frac{p_4}{q_4} < \frac{p_2}{q_2} = \frac{1}{a_1} < 1$$

5. $|q_n\xi - p_n| < |q_{n-1}\xi - p_{n-1}|$

The proof of the above lemma can be found in the Appendix.

Define $x_n = q_n\xi - p_n$ so then $x_n < 0$ if n is even and > 0 if n is odd. Then we have

$$x_{n+1} = a_n x_n + x_{n-1}.$$

and so

$$-1 = x_0 < x_2 < x_4 < \cdots < 0 < \cdots < x_5 < x_3 < x_1 = \xi < 1$$

Since

$$\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^n}{q_n q_{n+1}}$$

and ξ lies between p_n/q_n and p_{n+1}/q_{n+1} but closer to p_{n+1}/q_{n+1}, it follows that x_n lies between $\frac{(-1)^{n+1}}{2q_{n+1}}$ and $\frac{(-1)^{n+1}}{q_{n+1}}$ so the error

$$\frac{1}{2q_{n+1}} \leq |x_n| \leq \frac{1}{q_{n+1}}. \tag{28}$$

Definition 5.22. The sequence $\{\lambda, \lambda^2, \lambda^2, \dots\}$ is said to have a *close return to* 1 *at time* q if λ^q is closer to 1 than any previous iterations:

$$|\lambda^q - 1| < \left|\lambda^k - 1\right| \text{ for } k = 1, 2, \dots, q-1$$

We will primarily use the additive model of the circle $\mathbb{R}/\mathbb{Z}$. Using the isomorphism $\xi \longmapsto e^{2\pi i\xi}$, we can equivalently consider the dynamics of iterating

$$x \longmapsto x + \xi \mod \mathbb{Z}$$

and the orbit

$$0 \longmapsto \xi \longmapsto 2\xi \longmapsto 3\xi \longmapsto \dots$$

Definition 5.23. Define the *distance* function $\|\cdot\| : \mathbb{R}/\mathbb{Z} \longrightarrow [0, 1/2]$ as

$$\|x\| = \text{dist}\,(x, \mathbb{Z}) = \min\{|x - n| \mid n \in \mathbb{Z}\}.$$

With this, we can more concisely define $q \geq 1$ to be the *close return time* if $\|q\xi\| < \|k\xi\|$ whenever $0 < k < q$.

An elementary geometric argument gives,

$$\left|\lambda^k - 1\right| = 2\sin(\pi\|k\xi\|)$$

where the map $x \longmapsto 2\sin(\pi x)$ is strictly increasing for $0 \leq x \leq 1/2$. Using trivial bounds for the sin function, we have for $0 < k$,

$$4\|k\xi\| \leq \left|\lambda^k - 1\right| \leq 2\pi\|k\xi\| \tag{29}$$

(compare with (22)).

In the special case that $k = q_n$, we have $k\xi \equiv q_n\xi \equiv p_n + x_n \equiv x_n \mod \mathbb{Z}$ and if $n \geq 2$, $|x_n| < 1/2$ by (28). Thus $\|q_n\xi\| = |x_n|$.

Remark 5.24. Note that, for $a, b \in \mathbb{Z}$,

$$a\xi \equiv b\xi \mod \mathbb{Z} \Longleftrightarrow a = b.$$

Indeed $a\xi \equiv b\xi \mod \mathbb{Z} \Longleftrightarrow a\xi = b\xi + k$ for $k \in \mathbb{Z}$ and since ξ is irrational, the result holds.

Lemma 5.25. *For an integer m, $0 < m \leq q_{n+1}$, $m\xi$ has a representative $\mod \mathbb{Z}$ that lies strictly between x_{n-1} and $x_n \Longleftrightarrow m = q_{n-1} + jq_n$ for some integer $1 \leq j \leq a_n$.*

Proof. This proof is due to Milnor's lecture at Harvard (Milnor 2005). If n is odd, then $x_{n-1} < 0 < x_n < -x_{n-1}$. Then certainly

$$x_{n-1} < x_{n-1} + x_n < x_{n-1} + 2x_n < \cdots < x_{n-1} + a_n x_n = x_{n+1} < 0 < x_n \tag{30}$$

and so if $m = q_{n-1} + jq_n, 0 < j \leq a_n$, then $m\xi \equiv q_{n-1}\xi + jq_n\xi \equiv x_{n-1} + jx_n \mod \mathbb{Z}$. If n is even, an analogous argument holds where all inequalities are simply reversed. Hence, the right to left direction is true. For the converse, we do a double induction on n and m. Specifically, let $\mathbf{A}_n$ and $\mathbf{B}_n$ be the assertions

$\mathbf{A}_n$: No point $m\xi$ with $0 < m \leq q_n$ has a representative $\mod \mathbb{Z}$ which lies strictly between x_{n-1} and x_n

$\mathbf{B}_n$: For $0 < m \leq q_{n+1}$, the only representatives of $m\xi \mod \mathbb{Z}$ that lie strictly between x_{n-1} and x_n are those of the form $x_{n-1} + jx_n$ for $1 \leq j \leq a_n$.

We will prove that $\mathbf{A}_n \implies \mathbf{B}_n \implies \mathbf{A}_{n+1}$ and so if $\mathbf{A}_1$ is true, it follows that $\mathbf{A}_1 \implies \mathbf{B}_1 \implies \mathbf{A}_2 \implies \mathbf{B}_2 \implies \mathbf{A}_3 \implies \dots$ are all true by induction.

Certainly $\mathbf{A}_1$ is true, trivially. Suppose $\mathbf{B}_n$ is true. If n is odd, then we have

$$x_{n-1} < x_{n+1} < 0 < x_n < -x_{n-1} < -x_{n+1}$$

so if we have $m\xi$ with $0 < m \leq q_{n+1}$ with representative lying strictly between x_n and x_{n+1}, this representative would also lie between x_{n-1} and x_n. From $\mathbf{B}_n$, we know this representative is of the form $x_{n-1} + jx_n$ for some $1 \leq j \leq a_n$. However, all of these are $\leq x_{n+1}$ by (30) so we have a contradiction. An analogous argument holds if n is even where all inequalities are simply reversed. Thus $\mathbf{B}_n \implies \mathbf{A}_{n+1}\ \forall\, n$.

We now only need to verify that $\mathbf{A}_n \implies \mathbf{B}_n$. Fix $n \in \mathbb{N}$ and suppose $\mathbf{A}_n$ is true. We proceed by induction on m. Let $0 < m \leq q_{n+1}$ and suppose $m\xi$ has a representative $y_m \mod \mathbb{Z}$ lying strictly between x_{n-1} and x_n. From $\mathbf{A}_n$, we must have $m > q_n\, (> q_{n-1})$. Then $(m - q_n)\xi \equiv y_m - x_n \mod \mathbb{Z}$.

If y_m lies between $x_n + x_{n-1}$ and x_n, then $y_m - x_n$ will lie between x_{n-1} and 0. If further $y_m = x_n + x_{n-1}$, then $m = q_n + q_{n-1}$ by (5.24) and so the result holds. Moreover, as $m > q_n,\ \ y_m \neq x_n$. Thus $y_m - x_n$ lies *strictly* between x_{n-1} and 0.

Since $m - q_n < m$, we can apply the inductive hypothesis so $y_m - x_n$ is of the form $x_{n-1} + jx_n$ for some integer k. Hence, y_m is of the same form.

If y_m does not lie between x_n and $x_{n-1} + x_n$, y_m would be contained in the open interval between x_{n-1} and $x_{n-1}+x_n$ (e.g. by looking at (30)). Then $y_m - x_{n-1}$ would be strictly between 0 and x_n and so $(m-q_{n-1})\xi$ has a representative strictly between 0 and x_n. The inductive hypothesis would then suggest $y_m - x_{n-1} = x_{n-1} + jx_n$ for $0 < j \leq a_n$. However, if n is odd, this is always $< 0 < x_n$ by (30) and if n is even, this is always $> 0 > x_n$ be reversing (30). Hence, this case cannot happen.

Thus $\mathbf{B}_n$ holds for all n. So if m is as in B_n with representative y_m we have

$$y_m = x_{n-1} + jx_n, \ \ 0 < j \leq a_n$$

$$\Rightarrow m\xi \equiv (q_{n-1} + jq_n)\xi \mod \mathbb{Z}$$

$$\Rightarrow m = q_{n-1} + jq_n, \ \ 0 < j \leq a_n$$

with the last implication following from Remark (5.24). □

Theorem 5.26. *The point* $\lambda^q = e^{2\pi q i \xi}$ *is a closest return to 1 along the orbit*

$$1 \longmapsto \lambda \longmapsto \lambda^2 \longmapsto \lambda^3 \longmapsto \ldots$$

if and only if q *is one of the denominators* $1 \leq q_1 \leq q_2 < q_3 < \ldots$ *of the continued fraction of* ξ*. Moreover,*

$$\frac{2}{q_{n+1}} < |\lambda^n - 1| < \frac{2\pi}{q_{n+1}}.$$

Proof. The first part follows from assertion $\mathbf{A}_n$ in the above lemma and the second from (28) and (29). □

Since $q_{n+1} = a_n q_n + q_{n-1} > q_n + q_{n-1} > 2q_{n-1}$, we see that the close return times q_n increase at least exponentially as $n \longrightarrow \infty$ and so the close return distances decrease at least exponentially fast to 0 (from the bounds in Theorem 5.26).

5.7 Recent Developments, Problems and Open Conjectures

More recent developments in the field have revealed more sophisticated criterion on when local linearisation is possible. We state three of these without proof though proofs can be found in the relevant references in the bibliography.

Recall the definition of the n^{th} convergent to ξ:

$$\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_{n-1}}}}.$$

For the rest of this section, $\lambda = e^{2\pi i \xi}$ where $\xi \in \mathbb{R}/\mathbb{Z}$ and $\{q_n\}$ will be the sequence of denominators of the n^{th}

convergent.

Theorem 5.27. *(Brjuno 1965) If*

$$\sum_{n=1}^{\infty} \frac{\log(q_{n+1})}{q_n} < \infty \tag{31}$$

then any holomorphic germ with a fixed point of multiplier λ is locally linearisable.

With regards to non-linearisation, a more robust impediment to f being conjugate to a rotation, is the so called *small cycles property.*

Definition 5.28 (Small Cycles)**.** A fixed point is said to have the *small cycles property* if every neighbourhood of the fixed point contains infinitely many periodic orbits.

Intuitively, a fixed point has the small cycles property if it can be approximated by small cycles.

Jean-Christophe Yoccoz proved a partial converse to the theorem of Brjuno et. al, specifically showing that it is the best possible result for the quadratic maps $f(z) = z^2 + \lambda z$.

Theorem 5.29. *(Yoccoz, 1988) Conversely, if the sum in* (31) *diverges, then the quadratic function $f(z) = z^2 + \lambda z$ is not locally linearisable about the origin. Furthermore, the origin has the small cycles property.*

Yoccoz's Theorem raises the natural question as to whether every Cremer point necessarily has the small cycles property. This was answered only 30 years ago by Ricardo Perez-Marco who completely characterised multipliers that have the small cycles property.

Theorem 5.30. *(Perez-Marco, 1990). Suppose the sum in* (31) *diverges. Then, if*

$$\sum_{n=1}^{\infty} \frac{\log\log(q_{n+1})}{q_n} < \infty \tag{32}$$

any germ of a holomorphic function which has a Cremer point at the origin will exhibit the small cycles property. Conversely, if (32) *diverges then there exists a holomorphic function with the origin a fixed point of multiplier λ which is not linearisable but also does not have the small cycles property.*

Many results have been established in only the last 20 years. Notably, X. Buff and A. Chéritat formulated an upper bound on the size of quadratic Siegel discs, settling a conjecture on the bound of the so-called *Brjuno function* in 2003 (Buff and Chéritat 2004).

The proofs of the above are highly non-trivial. Without saying anything further on the above results, we consider a substantially weaker condition for the smalls cycle property.

Theorem 5.31. *(Problem 11-d in Milnor 2006). Suppose that, for $d \in \mathbb{Z}_{\geq 2}$*

$$\limsup_{q \longrightarrow \infty} \frac{\log\log\left(1/\left|\lambda^q - 1\right|\right)}{q} > \log d > 0 \tag{33}$$

Then any fixed point of multiplier λ for a degree d rational function has the small cycles property (and so is not locally linearisable).

Proof. It suffices to prove the theorem for the function

$$f(z) = z^d + \cdots + \lambda z$$

by applying the same reduction as in the proof of Cremer's Theorem (5.5). Fix $\delta > 0$ such that f is analytic throughout $\mathbb{D}_\delta$. As $f(0) = f'(0) = 0$, the function

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

is holomorphic throughout $\mathbb{D}_\delta$. Hence, by the maximum modulus principle,

$$|f(z)| \leq M\,|z|\,, \quad |z| < \delta$$

where $M = \sup_{z \in \mathbb{D}_\delta} |g(z)|$.

Proceeding in exactly the same way as in the proof of lemma (5.4), we see that if $|\lambda^q - 1| < 1$, we have a q-periodic point z_q where

$$0 < |z_q| < |\lambda^q - 1|^{1/(d^q - 1)} < |\lambda^q - 1|^{1/d^q}\,.$$

Let $\varepsilon = \varepsilon(\delta) > 0$ such that $e^\varepsilon > M$. Then, by assumption, for arbitrarily large q

$$\frac{\log\log\left(1/\left|\lambda^q - 1\right|\right)}{q} > \log(d) + \varepsilon$$

$$\Longrightarrow |\lambda^q - 1|^{1/d^q} < \exp(-e^{\varepsilon q}).$$

Now since $|f(z)| < e^{\varepsilon} |z|$ whenever $|z| < \delta$, we have

$$\left|f^{ok}(z)\right| < \delta \text{ for } 1 \leq k \leq q, \; \forall \; |z| < e^{-\varepsilon q}\delta.$$

Now we have a periodic point $|z_q|$ such that

$$0 < |z_q| < \exp\left(-e^{\varepsilon q}\right) < e^{-\varepsilon q}\delta$$

for large q. Thus, the periodic orbit $\mathcal{O}(z_q) = \{z_q, f(z_q), \ldots, f^{oq}(z_q)\} \subset \mathbb{D}_{\delta}$. As this holds for arbitrarily large q, we have infinitely many periodic orbits in $\mathbb{D}_{\delta}$. As δ was arbitrary, the orgin must have the small cycles property. □

There are still many unsolved problems related to local normal forms of analytic maps near fixed points. Most of these are related to the irrationally indifferent case given the rich behaviour and novelty of many of the results. For example, the following are, at the point of writing, open problems,

- Does a Julia set that contains a Cremer point always have a positive Lebesgue measure?
- (Smale's Mean Value Conjecture) Let f be any polynomial of the form $f(z) = z + a_2 z^2 + \cdots + a_d z^d$, does there exist a critical point c of f for which $\left|\frac{f(c)}{c}\right| \leq 1$? (Miles-Leighton and Pilgrim 2012)
- (Douady and Sullivan, 1980) Is the boundary of a Siegel disc of a rational map always a Jordan curve?
- Does there exist a germ of a rational function that is not locally linearisable but also does not have the small cycles property?

5.8 Siegel Discs and the Postcritical Closure

In the previous sections, we've shown that every attracting periodic orbit attracts a critical point (Theorem 2.11), and that every parabolic fixed point contains a critical point in each of its basins (Theorem 4.17). For an irrationally indifferent fixed point, there are again relations — albeit less direct — between the dynamics and the set of critical points.

Definition 5.32. The *postcritical closure* of a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the topological closure of the strict forward orbits of the critical points of f:

$$= (f) = \overline{\bigcup_{n>0} f^{\circ n}(V)}$$

in which $V = \{z \in \hat{\mathbb{C}} \mid fz = 0\}$.

Equivalently, (f) is the smallest closed set that contains the critical values of $f^{\circ n}$ for every $n > 0$.

We have the following result:

Theorem 5.33. *(f) contains the attracting and superattracting periodic orbits of f, the indifferent periodic orbits of f which lie in the Julia set, and the boundary of any domain on which f is conjugate to a rotation. In particular, it contains every periodic orbit which is parabolic or Cremer, as well as the boundary of every period of Siegel discs.*

One proof of this theorem proceeds by endowing the complement $Q = \hat{\mathbb{C}} \setminus (f)$ of the postcritical set with its Poincaré metric (this is possible so long as this complement is hyperbolic as a Riemann surface, which happens so long as (f) contains at least three points; in the case $|(f)| < 3$ it turns out that the map is always conjugate to $z \mapsto z^{\pm d}$ where $d \geq 2$ is the degree of f, and the dynamics of this exceptional case are well-understood.) It can then be demonstrated that the iterates $f^{\circ k}$ expand the distances on Q, and *strictly* expands distances at points in the Julia set whose forward orbits are disjoint from Q. These facts prohibit Q from containing attracting fixed points, or indifferent fixed points in the Julia set. With some modifications, a similar argument can be applied to yield the conclusion for the boundaries of rotation domains. The details of the proof depend on more machinery than can be exposited within the length of this report in a satisfactory manner; we refer instead to the references (McMullen, 1994 and Milnor, 2006).

Propositions such as Theorem 5.33 allow us to establish conclusions about the dynamics of the map by investigating the forward orbit of the critical set. (Nevertheless, we point out that there are cases where this is not very useful: for example, Rees (1942) demonstrates that there exist rational maps for which $= \hat{\mathbb{C}}$.)

One might also attempt to further one's intuition of the dynamics of a map by computing these forward orbits and observing their behaviour. In the following we present a collection of computer-generated images, produced by the authors of this report, which are visually indicative of features of rational maps which have been discussed.

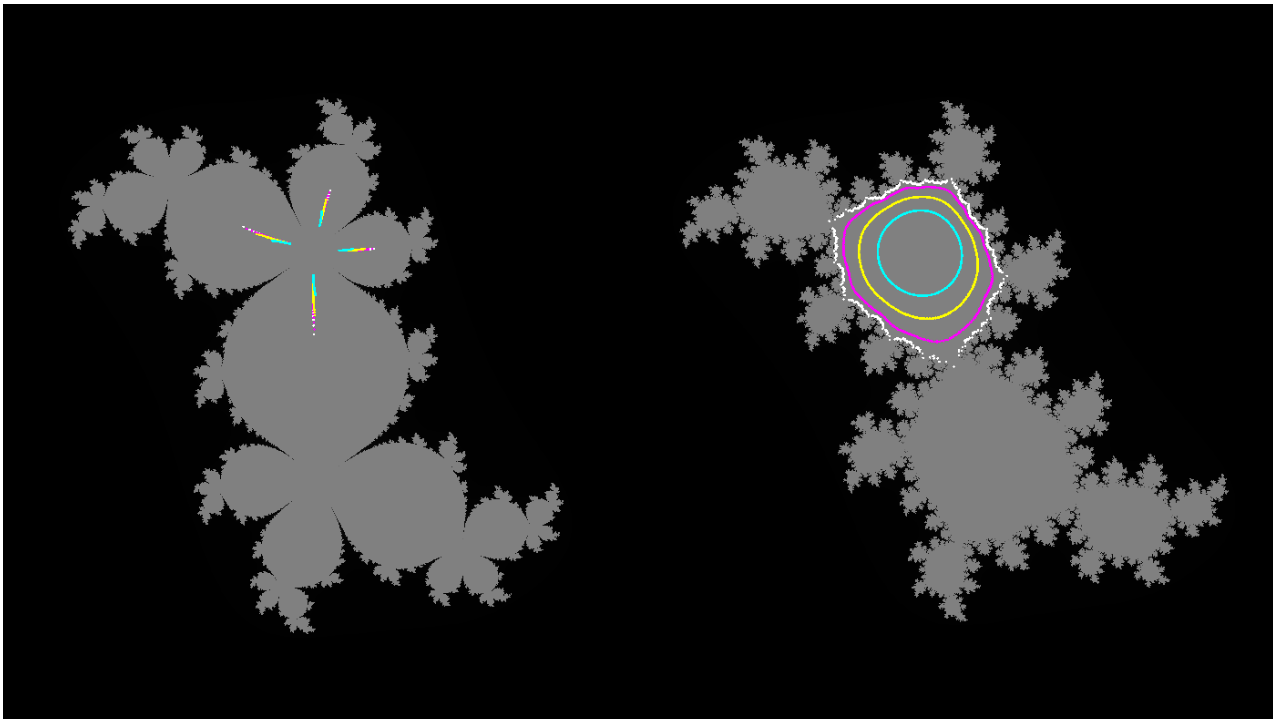

Figure 5: Filled Julia sets (grey) with the forward images of the critical point (white) and several nearby points (colours) plotted for a number of iterations. For $\xi \approx 0.2949$ (right), the forward orbit of the critical point delineates the boundary of a domain on which the dynamics are conjugated to a rotation, whereas for $\xi = 0.25$ (left) we observe convergence to a cycle of period 4.

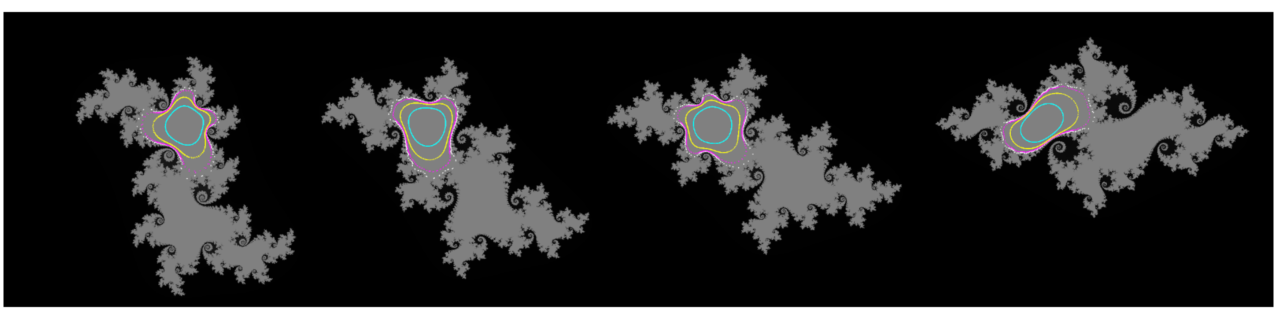

Figure 6: Filled Julia sets and selected orbits for $\xi \approx 0.2475, 0.3408, 0.4023,$ and 0.4892; the shape of the discs are suggestive of nearby rational numbers $(1/4, 1/3, 2/5, 1/2.)$

6 Appendix

6.1 Preliminary Results in Complex Analysis

Theorem 6.1. (J *Is not Empty, Milnor*). *If f is rational of degree 2 or more, then the Julia set $J(f)$ is nonvacuous.*

Theorem 6.2. *(Weierstrass Uniform Convergence Theorem, Milnor). If a sequence of holomorphic functions $f_n : U \to \mathbb{C}$ converges uniformly to the limit function f, then f itself is holomorphic. Furthermore, the sequence of derivatives f_n' con- verges, uniformly on any compact subset of U, to the derivative f'*

Theorem 6.3. *(Permanence Principle) For one variable, the principle of permanence states that if $f(z)$ is an analytic function defined on an open connected subset U of the complex numbers $\mathbb{C}$ and there exists a convergent sequence $\{a_n\}$ having a limit L which is in U, such that $f(a_n) = 0$ for all n, then $f(z)$ is uniformly zero on U.*

Theorem 6.4. *(Picard's Theorem, Milnor). Every holomorphic map $f : \mathbb{C} \to \mathbb{C}$ which omits two different values must necessarily be constant.*

Theorem 6.5. *(Cauchy Derivative Estimate). If f maps the disk of radius r about z_0 into some disk of radius s, then*

$$|f'(z_0)| \leq s/r$$

Theorem 6.6. *(Open Mapping Theorem, wikipedia) If $f : U \to \mathbb{C}$ is a non-constant holomorphic function, where $U \subset \mathbb{C}$, then f sends open subsets of U to open subsets of* $\mathbf{C}$.

Theorem 6.7. *If a sequence of univalent functions $\{f_n\}$ on a domain converges locally uniformly to f, then f is either constant or univalent.*

Lemma 6.8. *(Basin Boundary = Julia Set). If $\mathcal{A} \subset \widehat{\mathbb{C}}$ is the basin of attraction for some attracting periodic orbit, then the topological boundary $\partial\mathcal{A} = \overline{\mathcal{A}}\backslash\mathcal{A}$ is equal to the entire Julia set. Every connected component of the Fatou set $\widehat{\mathbb{C}}\backslash J$ either coincides with some connected component of this basin $\mathcal{A}$ or else is disjoint from $\mathcal{A}$.*

Theorem 6.9 (Classification of Fatou components)**.** *For any holomorphic $f : S \to S$ from a hyperbolic Riemann surface to itself, exactly one of the four cases is true:*

- Attracting. *f has an attracting fixed point, and all orbits converge locally uniformly towards it.*
- Escape. *No orbit in f has an accumulation point.*

- Finite order. *Some iterate $f^{\circ n}$ is the identity map (thus every point in S is periodic.)*
- Irrational rotation. *S is conformally isomorphic to a disc, a punctured disc, or an annulus, and f is conjugate to a rotation $z \mapsto e^{2\pi i \xi}$ with ξ irrational.*

6.2 Riemann Surfaces and the Riemann Sphere

Most of our results concern the behaviour near a fixed point. It makes sense then that many of them have generalisations to the setting of *Riemann surfaces,* which are objects which 'locally resemble' an open subset of $\mathbb{C}$. This notion is formalised as follows

Definition 6.10. A *Riemann surface* S is a topological space such that for any $p \in S$, there is a neighbourhood U of p and a map called the *local uniformising parameter*:

$$\Phi : U \mapsto \mathbb{C}$$

homeomorphically mapping U to an open subset of the complex plane. Furthermore, for any two such neighbourhoods U and U' with nonempty intersection and local uniformising parameters Φ and Ψ, $\Psi \circ \Phi^{-1}$ is a holomorphic function on $\Phi(U \cap U')$.

In the report, we make use of the following result about families of maps between Riemann surfaces:

Theorem 6.11 (Montel)**.** *Any family of holomorphic maps between hyperbolic Riemann surfaces is normal.*

Of particular interest to us is the *Riemann sphere*, denoted $\widehat{\mathbb{C}}$. As a set, $\widehat{\mathbb{C}}$ is equal to $\mathbb{C} \cup \{\infty\}$, the complex numbers together with infinity. We make $\widehat{\mathbb{C}}$ into a Riemann surface by defining a pair of local uniformising parameters, each omitting a point on the sphere:

$$\begin{aligned} \zeta_0 : \widehat{\mathbb{C}} \setminus \{\infty\} &\to \mathbb{C} \\ z &\mapsto z \\ \zeta_\infty : \widehat{\mathbb{C}} \setminus \{0\} &\to \mathbb{C} \\ 0 &\mapsto \infty \\ z &\mapsto \frac{1}{z} \end{aligned}$$

The transition maps are $(\zeta_\infty \circ \zeta_0^{-1})(z) = \frac{1}{z}$ and $(\zeta_0 \circ \zeta_\infty^{-1})(z) = \frac{1}{z}$, both of which are evidently holomorphic on $\widehat{\mathbb{C}} \setminus \{0, \infty\}$.

It's also of interest to consider the topology on the Riemann Sphere. The open set's are exactly the open sets of $\mathbb{C}$ and the sets of the form $U \cup \{\infty\}$, where $U \subset \mathbb{C}$ is such that $\mathbb{C} \backslash U$ is compact. $\widehat{\mathbb{C}}$ is said to be the one-point compactification of the complex plane into the a sphere, since $\widehat{\mathbb{C}}$ is compact and by adding ∞ the space is diffeomorphic to the unit sphere.

6.3 Results on Infinite Series and Products

We make use of several well-known results on seires and products, which are summarised here; for their proofs we refer to Knopp (1990).

Theorem 6.12. *For a $t \in \mathbb{R}$ such that $|t| < 1$,*

$$\sum_{j=0}^{\infty} \binom{j+k}{j} t^j = (1-t)^{-(k+1)}$$

Theorem 6.13. *Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers. Then if $\sum_{k \in \mathbb{N}} a_k$ converges absolutely, then so does $\prod_{k \in \mathbb{N}} (1 + a_k)$.*

Theorem 6.14. *Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers such that $\sum_{k \in \mathbb{N}} a_k$ converges. Suppose furthermore that $0 < a_k < 1$ for each a_k. Then $\prod_{k \in \mathbb{N}} (1 - a_k) > 0$.*

6.4 Continued Fractions

Let $\xi \in [0, 1)$ be an irrational number with continued fraction

$$\xi = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

Denote by

$$[a_1, a_2, a_3, \ldots, a_n] = \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}$$

Lemma 6.15. *If ξ has the above continued fraction decomposition, and p_n and q_n are integers defined by*

$p_0 = 1, p_1 = 0, q_0 = 0, q_1 = 1$ and

$$p_{n+1} = a_n p_n + p_{n-1}$$

$$q_{n+1} = a_n q_n + q_{n-1}$$

then $[a_1, a_2, a_3, \ldots, a_{n-1}] = \frac{p_n}{q_n}$ for $n \geq 2$

Proof. Proceed by induction on n. If $n = 2$, the result can be immediately verified. Now assuming

$$[a_1, a_2, \ldots, a_{n-1}] = \frac{p_n}{q_n} = \frac{a_{n-1}p_{n-1} + p_{n-2}}{a_{n-1}q_{n-1} + q_{n-2}}$$

we have

$$\begin{aligned}
[a_1, a_2, \ldots, a_{n-1}, a_n] &= \left[a_1, a_2, \ldots, a_{n-1} + \frac{1}{a_n}\right] \\
&= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-1} + p_{n-2}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-1} + q_{n-2}} \\
&= \frac{a_n(a_{n-1}p_{n-1} + p_{n-2}) + p_{n-1}}{a_n(a_{n-1}q_{n-1} + q_{n-2}) + q_{n-1}} \\
&= \frac{a_n p_n + p_{n-1}}{a_n q_n + q_{n-1}} \\
&= \frac{p_{n+1}}{q_{n+1}}
\end{aligned}$$

□

Hence $\{q_n\}$ is a strictly increasing, unbounded sequence of integers.

Corollary 6.16. *With p_n and q_n as above, we have*

$$p_n q_{n+1} - p_{n+1} q_n = (-1)^n$$

Proof. We can rewrite the recurrence relation in matrix form as

$$\begin{pmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix}$$

and so comparing determinants

$$p_n q_{n+1} - p_{n+1} q_n = (-1)(p_{n-1} q_n - p_n q_{n-1}).$$

The result now follows. □

Lemma 6.17. $\frac{p_n}{q_n} < \xi$ *if* n *is odd and* $\frac{p_n}{q_n} > \xi$ *if* n *is even*

Proof. Proceeding by induction again, we see that the result is immediate for $n = 1$ or 2. Now $[a_2, a_3, \ldots, a_{n-1}]$ is the n^{th} convergent of $\frac{1}{\xi} - a_1$. Then by the induction hypothesis, if n is even

$$[a_2, a_3, \ldots, a_{n-1}] < \frac{1}{\xi} - a_1$$

$$\Rightarrow \frac{p_n}{q_n} = \frac{1}{a_1 + [a_2, a_3, \ldots, a_{n-1}]} > \xi$$

The same argument holds for n odd where the inequalities are simply reversed. □

Corollary 6.18. *We have* $|q_n \xi - p_n| < |q_{n-1}\xi - p_{n-1}|$ *and*

$$0 = \frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_5}{q_5} < \cdots < \xi < \cdots < \frac{p_6}{q_6} < \frac{p_4}{q_4} < \frac{p_2}{q_2} = \frac{1}{a_1} < 1.$$

Proof. The second statement follows immediately from the first and the above lemma so it suffices to prove that $|q_n \xi - p_n| < |q_{n-1}\xi - p_{n-1}|$. Let $r_1 \in (0, 1)$ be such that $r_1 = 1/\xi - a_1$. Then define $r_{n+1} \in (0, 1)$ inductively such that $r_{n+1} = 1/r_n - a_{n+1}$. Then $\xi = [a_1, a_2, \ldots, 1/r_n]$. By applying Lemma 6.15 to $[a_1, a_2, \ldots, 1/r_n]$, we have

$$\xi = \frac{p_n + p_{n-1} r_n}{q_n + q_{n-1} r_n}.$$

Hence, $|q_n \xi - p_n| = r_n |q_{n-1}\xi - p_{n-1}|$ and $r_n < 1$. The result now follows since $q_n > q_{n-1}$. □

7 Acknowledgements

We would like to thank our supervisor Prof. Davoud Cheraghi for all his support and guidance throughout this project, which we have all thoroughly enjoyed compiling.

And if you liked our frighteningly small margins, thank Fermat. He was our inspiration.

References

[1] Beardon, Alan F. 2000. *Iteration Of Rational Functions.* New York, NY: Springer.

[2] Brjuno, Alexander D. 1965. *Convergence of transformations of differential equations to normal forms.* Dokl. Akad. Nauk USSR 165, 987-989 (Soviet Math. Dokl., 1536-1538).

[3] Buff, Xavier and Chéritat, Arnaud. 2004. *Upper Bound For The Size Of Quadratic Siegel Disks.* Inventiones Mathematicae 156 (1): 1-24. doi:10.1007/s00222-003-0331-6.

[4] Carleson, Lennart and Gamelin, Theodore W. 2013. *Complex Dynamics.* New York, NY: Springer.

[5] Cremer, Hubert. 1938. *Über die Häufigkeit der Nichtzentren*, Math. Ann. 115, no. 1, 573–580. MR MR15132

[6] Écalle, Jean. 1981. *Les fonctions resurgentes, volumes I–III.* Publ. Math. Orsay, France.

[7] Fernandez, Jaume A. 2012. *Siegel's Theorem: An Application Of KAM Theory To Complex Dynamics.* Presentation, Uppsala University, 2012.

[8] Fremlin, David. 2011. *Measure Theory.* Avaliable from: `https://www1.essex.ac.uk/maths/people/fremlin/mt.htm`. Accessed June 2020.

[9] Geyer, Lukas. 2016. *Topics In Mathematics Complex Dynamics.* Lecture Notes, 2016.

[10] Iliashenko, IU. S, and S Yakovenko. 2008. *Lectures On Analytic Differential Equations.* Providence, Rhode Island: American Mathematical Society.

[11] Knopp, Konrad. 1990. *Theory and Application of Infinite Series.* Dover Publications.

[12] McMullen, Curtis T. 1994. *Complex Dynamics and Renormalization.* Available from: `http://people.math.harvard.edu/ ctm/papers/home/text/papers/real/book.pdf`. Accessed June 2020.

[13] Miles-Leighton, Hayley, and Pilgrim, Kevin M. 2012. *Smale's Mean Value Conjecture And Complex Dynamics*. Computational Methods And Function Theory 12 (2): 559-563. doi:10.1007/bf03321844.

[14] Milnor, John W. 2005. *Dynamics In One Complex Variable*. Presentation, Institute for Mathematical Sciences, SUNY, Stony Brook NY.

[15] Milnor, John W. 2006. *Dynamics In One Complex Variable*. Princeton, N.J.: Princeton University Press.

[16] Perez-Marco, Ricardo. 1990. *Sur la dynamique des germes de diffeomorphismes holomorphes de* $(\mathbb{C}, O)$ *et des diffcomorphismes analytiques du cercle* These, Paris-Sud.

[17] Rees, M. 1942. *Ergodic rational maps with dense critical point forward orbit.* Ergod. Th. & Dynam. Sys.4, 311–322.

[18] Siegel, Carl L. 1942. *Iteration of analytic functions*, Ann. of Math. (2) 43, 607–612. MR 7044

[19] Stoll, Danny. 2020. *A Brief Introduction To Complex Dynamics*. University of Chicago. Accessed June 8.

[20] Yoccoz, Jean-Christophe. 1988. *Linearieation des germes de diffeomorphismes holomorphes de* $(\mathbb{C}, 0)$, C. R. Acad. Sci. Paris 306, 55-58.