Local Normal Forms of Analytic Maps Near Fixed Points

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Outline

- $f : \mathcal{N} \longrightarrow \mathbb{C}$ analytic function
- $z^* \in \mathcal{N}$ fixed point $f(z^*) = z^*$
- Forward orbit

$$\mathcal{O}(z) := \left\{ z, f(z), f^{\circ 2}(z), \dots, f^{\circ n}(z), \dots \right\}$$

$$f^{\circ n} = \underbrace{f \circ f \circ \dots f \circ f}_{n}$$

- Conjugating by local biholomorphism ϕ

$$\phi \circ f \circ \phi^{-1}$$

- Since conjugation preserves dynamics, we assume fixed point is at the origin
- $\lambda = f'(0)$ is called the multiplier and is invariant under conjugation
- $z \in \mathcal{N}$ is a *periodic point with period q* of f if $f^{\circ q}(z) = z$ and $f^{\circ(q+1)}(z) \neq z$

Geometrically Attracting or Repelling Fixed Points

Definition.

The fixed point z^* of f is called *topologically attracting* if $\exists$ a neighbourhood U on which the iterates $f^{\circ n}$ are defined and converge uniformly to the constant map $z \longmapsto z^*$.

Topologically attracting and multiplier

Theorem.

Consider the function $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots$ *with fixed point* $z^* = 0$. *Then the origin is topologically attracting iff* $|\lambda| < 1$.

Koenigs linerisation

Theorem.

For $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$ such that $|\lambda| \notin \{0, 1\}$, there exists a local biholomorphic function $\hat{z} = \phi(z)$ in some neighbourhood $\mathcal{N}$ of 0 such that the following diagram commute and $\phi(0) = 0$

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\ f\ } & f(\mathcal{N}) \\ \Big\downarrow \phi & & \Big\downarrow \phi \\ \mathbb{C} & \xrightarrow[\lambda]{} & \mathbb{C} \end{array}$$

Here $\phi(z) = \lim_{n \longrightarrow \infty} f^{\circ n}(z)/\lambda^n$

Basin of attraction

Definition.

The *attraction basin* $\mathcal{A}$ of a fixed point z^* is the set of all points that converge to z^* under iterations of f

$$\mathcal{A}(z^*) = \{z_0 \mid \lim_{n\to\infty} f^{\circ n}(z_0) = z^*\}$$

The *immediate basin* $\mathcal{A}_0$ is the connected component of $\mathcal{A}$ that contains z^*.

Global linearisation for a geometrically attracting fixed point

Theorem (Global linearisation)**.**

Up to multiplication by a non-zero constant, there exists a unique local biholomorphic map $\phi : \mathcal{A} \to \mathbb{C}$ *such that the following diagram commute and* $\phi(0) = 0$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{A} \\ \downarrow \phi & & \downarrow \phi \\ \mathbb{C} & \xrightarrow[\lambda]{} & \mathbb{C} \end{array}$$

We can prove this by defining $\phi(z) = \lambda^{-n}\phi_r(f^{\circ n}(z))$ where n is the smallest integer for which $|f^{\circ n}(z)| < r$.

Inverse of linearization and critical point

In this lemma consider f is a rational function with degree ≥ 2 over $\widehat{\mathbb{C}}$ and the fixed point $z^* \in \mathbb{C}$ so that the local behaviour is exactly the same as before.

Lemma.

The local inverse in the last theorem $\psi_\varepsilon : \mathbb{D}_\varepsilon \to \mathcal{A}_0$ can be uniquely analytically extended to some maximal open disc $\mathbb{D}_r$ as $\psi_r : \mathbb{D}_r \to \mathcal{A}_0$ with $\psi_r(0) = 0$ and $\phi \circ \psi_r(\hat{z}) = \hat{z}$.

Furthermore, ψ_r can be continuously extended to the boundary $\partial\mathbb{D}_r$ and there exists at least one critical point of f in the $\psi_r(\partial\mathbb{D}_r)$.

Attracting periodic orbit

Definition.

A *periodic orbit* is an orbit $z_0 \to z_1 \to z_2 \to \cdots$ such that $z_m = f^{\circ m}(z_0) = z_0$ for some integer m. A periodic orbit is called *attracting* if the derivative $\left|(f^{\circ m})'(z_k)\right| < 1$.

Note: $\left|(f^{\circ m})'(z_k)\right|$ is the same for all z_k.

Definition.

Since each z_k is a fixed point of $f^{\circ m}$, they have corresponding immediate basins. The immediate basin $\mathcal{A}_0(\mathcal{O}, f)$ of a periodic orbit $\mathcal{O}$ is the union of the immediate basins of each point in the orbit under $f^{\circ m}$.

Attracting periodic orbit and critical point

Theorem.

For f a nonlinear rational map, the immediate basin of every attracting periodic orbit contains at least one critical point.

Idea of proof:

- $f^{\circ m}$ maps $\mathcal{A}^0(z_j)$ into itself.
- f has no critical point $\implies f^{\circ m}$ no critical point.
- Basin of attraction of a attracting fixed point must contain a critical point.

Approximating attracting periodic orbit

Corollary.

Such a rational map f has only finitely many attracting periodic orbits.

A logarithm for approximating periodic orbit:

- Locate all critical points of the function
- Iteratively apply the function from the critical point.
- Observe if it converges to a periodic orbit.

Note: May fail for large period. e.g.$f(z) = z^2 - 1.5$.

Topologically Repelling

Definition.

The fixed point z^* of f is called *topologically repelling* if for some neighbourhood $\mathcal{N}$ of z^*, $\forall z \in \mathcal{N}$ and $z \neq z^*$, $\exists\, n \in \mathbb{N}$ s.t. $f^{\circ n}(z)$ leaves $\mathcal{N}$. Here we call $\mathcal{N}$ a *forward isolating neighbourhood* of z^*.

Note: The only orbit that stays in $\mathcal{N}$ is the orbit of the fixed point z^*.

Topologically repelling and multiplier

Theorem.

Consider the function $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$ *with fixed point* $z^* = 0$. *Then the origin is topologically repelling iff* $|\lambda| > 1$.

generlization of ψ_ϵ in repelling case

Theorem.

For a repelling fixed point of f, there exists an entire bijective function ψ such that $\psi(0) = 0$ and ψ conjugates f to the linear map $\hat{z} \mapsto \lambda\hat{z}$. Moreover, ψ is unique (up to multiplication by a non-zero constant).

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C} \\ \psi \uparrow & & \psi \uparrow \\ \mathbb{C} & \xrightarrow[\lambda]{} & \mathbb{C} \end{array}$$

Idea of proof:

Choose the smallest n such that $z/\lambda^n \in \mathbb{D}_\varepsilon(0)$, then define $\psi(z) = f^{\circ n}(\psi_\varepsilon(z/\lambda^n))$.

Superattracting Fixed Points

Definition.

A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ has a super-attracting fixed point at $z^* \in \mathbb{C}$, if $f(z^*) = z^*$ and $f'(z^*) = 0$.

WLOG fixed point at 0, we can write:

$$f(z) = a_p z^p + a_{p+1} z^{p+1} \cdots = \sum_{k=p}^{\infty} a_k z^k \tag{1}$$

Bottcher's Theorem

Theorem.

Let f be as in Eq. 1. Then there exists a local holomorphic change of coordinates $\hat{z} = \phi(z)$, such that $\phi(0) = 0$ and $\phi'(0) = 1$, where $\phi(f(z)) = \phi(z)^p$ locally. Furthermore, ϕ is unique up to multiplication by a $(p-1)^{th}$ root of unity.

- existence and uniqueness of map $\phi : \mathbb{C} \to \mathbb{C}$
- $\phi(f(z)) = \phi(z)^p$
- derivative at 0 is 1
- existence of inverse ψ
- $\phi \circ f \circ \phi^{-1}(z) = z^p$

Inverse of the change of coordinates

Theorem.

Let f and ϕ be as in Theorem 3.1 and let ψ_r be local inverse of ϕ. Then there exists a unique open disc $\mathbb{D}_\varepsilon$ around 0 of maximal radius $0 < \varepsilon \leq 1$ such that ψ_r extends holomorphically to a map ψ from the disc into the immediate basin $\mathcal{A}^0$ of 0. If $\varepsilon = 1$, then ψ maps the unit disc biholomorphically onto $\mathcal{A}^0$ and 0 is the only critical point of f in the basin. On the other hand if $\varepsilon < 1$ then there is at least one other critical point of f in $\mathcal{A}^0$, lying on the boundary of $\psi(\mathbb{D}_\varepsilon)$.

- Holomorphic extension of ψ, to:
  1. Another critical point, ψ valid on $\mathbb{D}_\varepsilon$, critical point on boundary of image of $\mathbb{D}_\varepsilon$ under the map ψ.
  2. No other critical point, ψ biholomorphism from $\mathbb{D}_1$ to immediate basin.

First Example

Example.

Take:

$$f(z) = \frac{z^2}{1-2z^2} \approx z^2 + 2z^4 + 4z^6 \dots$$

By our Theorem 3.2 the extension of the inverse of the map valid in whole of $\mathbb{D}_1$. Inverse given by:

$$\psi(\hat{z}) = \frac{\hat{z}}{1+\hat{z}^2}$$

We then see:

$$f(\psi(\hat{z})) = f\left(\frac{\hat{z}}{1+\hat{z}^2}\right) = \frac{\hat{z}^2}{1+\hat{z}^4} = \psi(\hat{z}^2)$$

Applications to Polynomial Dynamics

Now we will be working on the Riemann Sphere, let:

$$f(z) = a_d z^d + a_{d+1} z^{d+1} \cdots + a_1 z + a_0 \quad (2)$$

- WLOG we can assume f to be monic
- super-attracting fixed point at ∞

Fixed point at ∞

Let $\zeta = \frac{1}{z}$. Then:

$$G(\zeta) = \frac{1}{f(1/\zeta)}$$

Then since f is monic, near ∞, $f(z) \approx z^d$. By that we have near 0,

$$G(\zeta) \approx \frac{1}{z^d} = \zeta^d$$

Then from Theorem 3.1 we can get a map α, which conjugates G to $\hat{z} \mapsto \hat{z}^d$. Let:

$$\phi(z) = \frac{1}{\alpha(\frac{1}{z})}$$

which maps some neighbourhood of ∞ biholomorphically onto another neighbourhood of ∞. We then have:

$$\phi(f(z)) = \phi(z)^d$$

Example

Example.

Let's take the map:

$$f(z) = z^2 - 2$$

Super-attracting fixed point at ∞. Let $\zeta = 1/z$ and get map $G(\zeta)$:

$$G(\zeta) = \frac{1}{f\left(\frac{1}{\zeta}\right)} = \frac{\zeta^2}{1 - 2\zeta^2}$$

For G we had the local inverse $\beta(\hat{z}) = \frac{\hat{z}}{1+\hat{z}^2}$, here we have:

$$\psi(\hat{z}) = \frac{1}{\beta(1/\hat{z})} = \hat{z} + \frac{1}{\hat{z}}$$

For verification we find:

$$f\left(\hat{z} + \frac{1}{\hat{z}}\right) = \hat{z}^2 + \frac{1}{\hat{z}^2} = \psi(\hat{z}^2)$$

Parabolic Fixed Points

Attraction vectors

$$f(z) = \lambda z + \mu z^{p+1} + \dots$$

- When $\lambda = 1$, fixed point exhibits both attractive and repulsive properties.
- Consider wanting for $\alpha \in \mathbb{R}^+$ (*non-constant*), $f(\varepsilon) = \alpha\varepsilon$, i.e. an infinitesimal vector on which f acts as scaling.
- For $\lambda = 1$, this has solutions $\varepsilon^p = (\alpha - 1)/\mu$
  - *attraction vectors* $v_-^p = -1/(p\mu)$
  - *repulsion vectors* $v_+^p = +1/(p\mu)$
- $v_j = v_0 \exp(j/p \cdot \pi i)$ repulsion for even j, attraction for odd j.

Attraction vectors

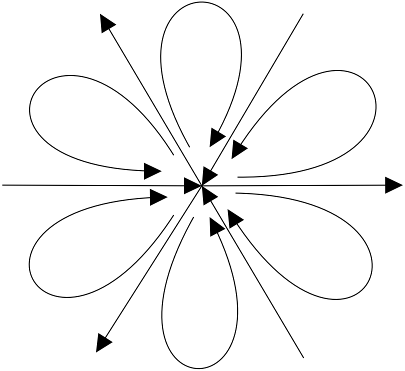

Figure: Attraction and repulsion vectors, basins where $p = 3$, $\mu \in \mathbb{R}^{>0}$

Attraction vectors

This intuition is formalized in the following results.

Theorem.

Let f be a holomorphic function as with $\lambda = 1$. Let z_0 be such that the sequence $z_n = f^{\circ n}(z_0) \longrightarrow 0$ but $\forall n, z_n \neq 0$. Then, for some attraction vector v_j satisfying $v_j^p = -1/(p\mu)$,

$$\lim_{n \longrightarrow \infty} n^{1/p} z_n = v_j$$

i.e. $z_n \sim v_j/n^{1/p}$ asymptotically. z_n is said to tend to 0 in the direction of v_j.

Corollary.

Let z_0 be such that the sequence $z_n = \left(f^{-1}\right)^{\circ n}(z_0) \longrightarrow 0$ but $\forall n, z_n \neq 0$. Then, for some repulsion vector v_j of f satisfying $v_j^p = 1/(p\mu)$, $z_n \sim v_j/n^{1/p}$.

Rotational parabolic points

We have sneakily been setting $\lambda = 1$. The following result justifies "WLOG $\lambda = 1$".

Definition.

Let v be a complex number.

- If there exists a sequence $z_n = f^{\circ n}(z_0) \longrightarrow 0$ (but $\forall n, z_n \neq 0$) with a subsequence z_{n_k} such that $\arg z_{n_k} \longrightarrow \arg v$, then v is called an attraction vector for f.
- If there exists a sequence $z_n = \left(f^{-1}\right)^{\circ n}(z_0) \longrightarrow 0$ (but $\forall n, z_n \neq 0$) with subsequence z_{n_k} such that $\arg z_{n_k} \longrightarrow \arg v$, then v is called an repulsion vector for f.

Theorem.

The attraction vectors of f are the same as the same as those of $f^{\circ r}$, and their number is a multiple of r.

Attraction basins

- A specialized “directional” notion of an attraction basin is thus needed for our purposes.
- Similarly, the notion of a *petal* acts as a directional notion of a neighbourhood of a fixed point.

Definition.

The basin of attraction $\mathcal{A}_v$ for an attraction vector v is defined as the set of points z such that $f^{\circ n}(z) \longrightarrow 0$ in the direction of v. The immediate basin of attraction $\mathcal{A}_v^0$ is defined as the unique connected component of $\mathcal{A}_v$ that is closed under f.

Definition.

Where f is injective on some neighbourhood $\mathcal{N}$ of its fixed point, an open set $\mathcal{P} \subseteq \mathcal{N}$ is called an attracting petal for f along attraction vector v if

1. $\mathcal{P}$ is closed under f.
2. $\mathcal{P} \subseteq \mathcal{A}_v$
3. Any orbit $f^{\circ n}(z_0)$ converging to 0 along v is eventually in $\mathcal{P}$.

Basins and petals

Basic expected results on attraction basins and neighbourhoods transfer to our new definitions.

Lemma.

The attraction basin is open.

Lemma.

The basins of attraction $\mathcal{A}_v$ are contained in the Fatou set of f, while their boundaries $\partial\mathcal{A}_v$ are contained in the Julia set.

Lemma.

Where f is a non-linear rational map with parabolic fixed point 0 and multiplier $\lambda = 1$:

1. *each immediate basin of 0 contains at least one critical point of f.*
2. *each basin contains exactly one petal $\mathcal{P}_{\max}$ that maps injectively onto some right half-plane under ϕ and is maximal with respect to this property.*
3. *$\mathcal{P}_{\max}$ has at least one critical point of f on its boundary.*

Abel linearization

- $\phi(f(z)) = \phi(z)$ is *not* a local homeomorphism!
- Better idea to find a linearization: take inspiration from the "inherent structure on the petal". $\phi(f(z)) = \phi(z) + 1$.

Theorem (Parabolic linearisation theorem)**.**

Given an attracting or repelling petal $\mathcal{P}$, there exists a unique (up to composition on the left with translation) conformal embedding $\phi : \mathcal{P} \to \mathbb{C}$ called a Fatou co-ordinate on $\mathcal{P}$ such that, for all $z \in \mathcal{P} \cup f^{-1}(\mathcal{P})$, we have:

$$\phi(f(z)) = \phi(z) + 1$$

Abel linearization

- Does not immediately suffice for a normal form
- Can "paste" linearization of each petal together – Ecalle-Voronin classification

Irrationally Indifferent Fixed Points

Local Linearisation

$\lambda = e^{2\pi i\xi}$ where $\xi \in [0, 1)$ is irrational.

Definition (Locally linearisable)**.**

The function f above is said to be *locally linearisable* if there is a local biholomorphic map ψ which conjugates f to a linear map:

$$\left(\psi^{-1} \circ f \circ \psi\right)(z) = \lambda z, \tag{3}$$

for all z in some neighbourhood of the origin.

We say an irrationally indifferent fixed point is a *Cremer point* if there is no local linearisation of f around the fixed point. A connected component of the Fatou set on which f is conjugate to a rotation of the unit disc is called a *Siegel disc*.

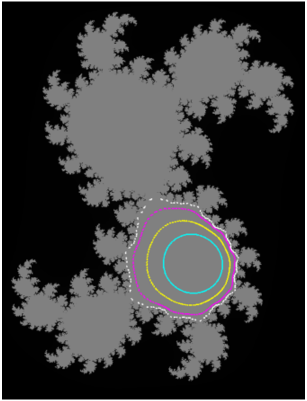

Figure: Example of Siegel Disc, in white, in filled Julia set. Here, the cyan, yellow and magenta depict orbits of points nearby the origin

Main Theorems:

- Cremer's Non-Linearisation Theorem
- Siegel's Linearisation Theorem
- Postcritical Closure

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Cremer's Non-linearization Theorem

Theorem.

(Cremer, 1938) Given $\lambda \in \mathbb{C}$ on the unit circle and $d \geq 2$, if the sequence $\sqrt[d^q]{1/|\lambda^q - 1|}$ is unbounded as $q \longrightarrow \infty$, no fixed point with multiplier λ of a rational function of degree d can be locally linearisable.

Sketch Proof Cremer's Theorem

Case when

$$f(z) = z^d + \cdots + \lambda z, d \geq 2$$

- $f^{oq}(z) = z^{d^q} + \cdots + \lambda^q z$
- Fixed points of f^{oq} satisfy the polynomial $z^{d^q} + \cdots + (\lambda^q - 1)z = 0$. Then

$$\prod_{j=1}^{d^q-1} |z_q(j)| = |\lambda^q - 1|$$

- $|\lambda^q - 1| < 1 \Longrightarrow \exists\, j_q \text{ s.t. } 0 < |z_q(j_q)| < |\lambda^q - 1|^{1/d^q}$
- Sequence $(q_k)_{k \geq 1}$ where

$$|\lambda^{q_k} - 1|^{-1/d^{q_k}} \longrightarrow \infty$$
$$\Longrightarrow |\lambda^{q_k} - 1|^{1/d^{q_k}} \longrightarrow 0$$

- Every neighbourhood of the origin has infinitely many periodic points

Siegel's Linearization Theorem

Definition.

For $\xi \in \mathbb{R}$, we say ξ is *Diophantine of order* $\leq \kappa$ if $\exists\, \varepsilon > 0$ such that

$$\left|\xi - \frac{p}{q}\right| > \frac{\varepsilon}{q^\kappa}$$

for any rational $\frac{p}{q}$

Certainly Diophantine of order $\leq \kappa \implies$ Diophantine of order $\leq \kappa + 1$

Lemma.

With ξ as above, ξ is Diophantine of order $\leq \kappa \iff$ there exists $M > 0$ such that $\forall\, q \in \mathbb{Z}_{\geq 1}$

$$1/\left|\lambda^q - 1\right| < M q^{\kappa - 1}$$

Theorem.

If the angle ξ is Diophantine of any order, then any holomorphic germ with multiplier $\lambda = e^{2\pi i \xi}$ is locally linearisable. Hence, if there exists $M > 0$ and $k \in \mathbb{N}$ such that $\forall\ q \in \mathbb{Z}_{\geq 1}$

$$1/\left|\lambda^q - 1\right| < Mq^k$$

then any holomorphic function with a fixed point of multiplier λ is locally linearisale.

Corollary.

In terms of the Lebesgue measure on $[0, 1)$, almost every ξ has the property that any holomorphic function with fixed point of multiplier $e^{2\pi i \xi}$ is locally linearizable.

Generic vs Lebesgue Almost Everywhere

- Cremer: for a *generic* choice of angle ξ, there exists a holomorphic function with fixed point of multiplier $\lambda = e^{2\pi i \xi}$ which is not locally linearizable.
- Siegel: for *almost every* angle ξ, any holomorphic function with fixed point of multiplier $\lambda = e^{2\pi i \xi}$ is locally linearizable.

A linguist would be shocked to learn that if a set is not closed this does not mean that it is open, or again that "E is dense in E" does not mean the same thing as "E is dense in itself"

- John Edensor Littlewood (1885–1977)

Siegel's Linearisation Theorem

Theorem (Siegel 1942)**.**

If ξ is Diophantine of any order, then every germ of a holomorphic map with fixed point of multiplier $\lambda = e^{2\pi i\xi}$ is locally linearisable.

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Theorem (Quadratic Siegel discs exist)**.**

For Lebesgue-almost all irrational $\lambda \in \mathbb{R}/\mathbb{Z}$, $f_\lambda(z) = \lambda z + z^2$ has a Siegel disc about the origin.

This is strictly weaker than Siegel (1942).

Theorem (Riemann mapping theorem)**.**

Let $U \subset \mathbb{C}$ be a simply connected domain. Then for every $z_0 \in U$ there is a unique conformal isomorphism $\phi : U \to \mathbb{D}$ to the unit disc such that

$$\phi(z_0) = 0 \quad \textit{and} \quad \phi'(z_0) > 0$$

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Definition (Conformal radius)**.**

The *conformal radius* of U as viewed from $z_0 \in U$ is $\operatorname{rad}(U, z_0) = \dfrac{1}{\phi'(z_0)}$.

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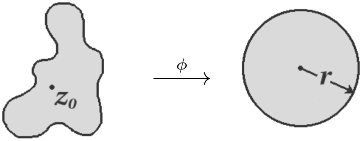
Definition (Conformal radius)**.**

The *conformal radius* of U as viewed from $z_0 \in U$ is $\mathrm{rad}(U, z_0) = \dfrac{1}{\phi'(z_0)}$.

Intuition: requiring instead $\phi : U \to \mathbb{D}_r$ with $\phi'(0) = 1$:



Definition (Conformal radius function)**.**

For $\lambda \in \mathbb{C}$, let $\sigma(\lambda)$ be the conformal radius from 0 of the maximal neighbourhood about 0 on which f_λ is conjugate to a rotation.

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If no such neighbourhood exists, set $\sigma(\lambda) = 0$.

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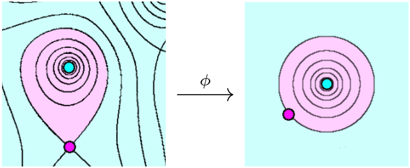
If no such neighbourhood exists, set $\sigma(\lambda) = 0$.

Some properties:

- σ is non-constant ($\sigma(0) = 0$, $\sigma(\lambda) > 0$ for $|\lambda| \notin \{0, 1\}$.)
- σ is *upper semi-continuous* on $\overline{\mathbb{D}}$.

- When $|\lambda| < 1$, Koenigs linearisation:



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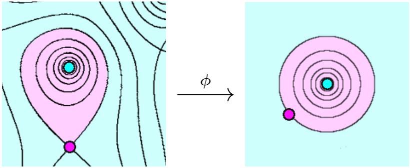

- $\eta(\lambda) = \phi(-\lambda/2)$. The function $\eta(\lambda)$ is holomorphic on the punctured disc. The singularity at the origin is removable — have $\sigma(\lambda) = |\eta(\lambda)|$ on $\mathbb{D}$.

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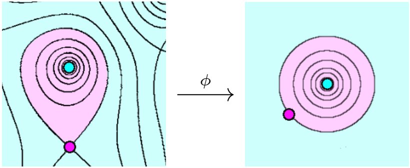

- $\eta(\lambda) = \phi(-\lambda/2)$. The function $\eta(\lambda)$ is holomorphic on the punctured disc. The singularity at the origin is removable — have $\sigma(\lambda) = |\eta(\lambda)|$ on $\mathbb{D}$.

- Computation:

$$\eta(\lambda) = -\frac{\lambda}{4} + \frac{\lambda^2}{16} + \frac{\lambda^3}{16} + \frac{\lambda^4}{32} + \frac{9\lambda^5}{256} + \frac{\lambda^6}{256} + \frac{7\lambda^7}{256} + O(\lambda^8)$$

Lemma (F. and M. Riesz, 1916)**.**

Let $\eta : \mathbb{D} \to \mathbb{C}$ be bounded and holomorphic. If for some constant $c \in \mathbb{C}$ the set of ξ such that

$$\lim_{r \nearrow 1} \eta(re^{2\pi i \xi}) = c$$

has positive Lebesgue measure, then η is constant.

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has positive Lebesgue measure, then η is constant.

Proof (quadratic Siegel discs exist).

$$\{\xi \in \mathbb{R}/\mathbb{Z} \mid z \mapsto e^{2\pi i\xi}z + z^2 \text{ not linearisable at } z = 0\}$$

$$= \{\xi \in \mathbb{R}/\mathbb{Z} \mid \sigma(e^{2\pi i\xi}) = 0\} \qquad \text{(by definition of } \sigma\text{)}$$

$$= \{\xi \in \mathbb{R}/\mathbb{Z} \mid \lim_{r \nearrow 1} \eta(re^{2\pi i\xi}) = 0\} \qquad \text{(upper semi-continuity)}$$

is a set of Lebesgue measure zero. □

Siegel's Linearisation Theorem

Strong version.

Theorem (Siegel 1942)**.**

If ξ is Diophantine of any order, then every germ of a holomorphic map with fixed point of multiplier $\lambda = e^{2\pi i\xi}$ is locally linearisable.

Schröder's equation: given f, seek ψ such that $f(\psi(z)) = \psi(\lambda z)$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\quad f \quad} & \mathbb{C} \\ \psi \Big\uparrow & & \Big\uparrow \psi \\ \mathbb{D}_r & \xrightarrow[w \mapsto \lambda w]{} & \mathbb{D}_r \end{array}$$

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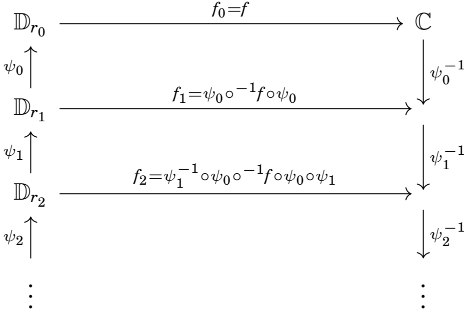
$$\Psi_0(z) = \sum_{j=2}^{\infty} \frac{b_j}{\lambda^j - \lambda} z^j$$

in which b_j are coefficients of the series expansion $F(z) = \sum_{j=2}^{\infty} b_j z^j$.

$$
\begin{array}{ccc}
\mathbb{D}_{r_0} & \xrightarrow{\quad f_0 = f \quad} & \mathbb{C} \\
\Big\uparrow {\scriptstyle \psi_0} & & \Big\downarrow {\scriptstyle \psi_0^{-1}} \\
\mathbb{D}_{r_1} & \xrightarrow{\quad f_1 = \psi_0 \circ^{-1} f \circ \psi_0 \quad} &
\end{array}
$$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ψ_2^{-1}

$$
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\psi_1 \uparrow & & \downarrow \psi_1^{-1} \\
\mathbb{D}_{r_2} & \xrightarrow{\quad f_2=\psi_1^{-1}\circ\psi_0\circ^{-1} f\circ\psi_0\circ\psi_1 \quad} & \\
\psi_2 \uparrow & & \downarrow \psi_2^{-1} \\
\vdots & & \vdots
\end{array}
$$

(hopefully)

$$
\mathbb{D}_{r_\infty} \xrightarrow{\quad z\mapsto\lambda z \quad} \mathbb{D}_{r_\infty}
$$

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- Carefully check:
  - Each ψ_k, ψ_k^{-1} well-defined.
  - $r_\infty > 0$.

What do Siegel discs *look like?* Compare results from previous sections:

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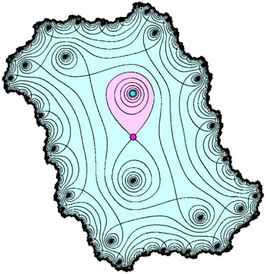
  - at least one critical point in basin of attraction.
  - at least one critical point on boundary of maximal linearising domain.



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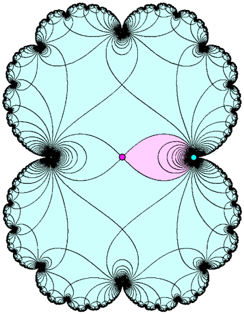

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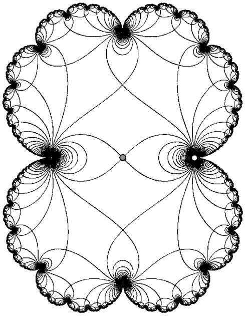

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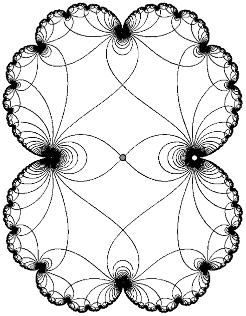
- *Irrationally indifferent fixed point*:
  - are there critical points on the boundary of Siegel discs? *Sometimes.*†



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† Herman M. 1985. *Are there critical points on the boundaries of singular domains?* Comm. Math. Phys., 99(4):593–612

Definition.

The *postcritical closure* of f is the topological closure of the forward orbit of the set of critical points:

$$P(f) = \overline{\bigcup_{k>0} f^{\circ k}(V(f))}$$

where

$$V(f) = \{z \in \hat{\mathbb{C}} \mid f'(z) = 0\}$$

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Equivalently, $P(f)$ is the smallest forward-invariant closed set containing all critical values of f.

Theorem.

Each of the following sets is contained within $P(f)$:

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- *all indifferent periodic orbits which lie within $\mathcal{J}(f)$.*
- *the boundary of every period of rotation domains.*

Sketch of proof. If $|P(f)| < 3$: special case. f is conjugate to $z \mapsto z^{\pm d}$.

Assume $|P(f)| \geq 3$. Equip $Q = \hat{\mathbb{C}} \setminus P(f)$ with the *Poincaré metric;* apply the Schwarz-Pick lemma to show that $f^{\circ k}$ expands distances on Q. □

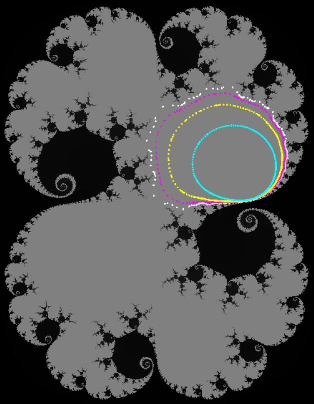

Figure: Filled Julia set (grey) for $\xi \approx 0283$ with forward orbit of critical point (white) together with several other points (magenta, yellow, cyan). Critical orbit delineates rotation domain.

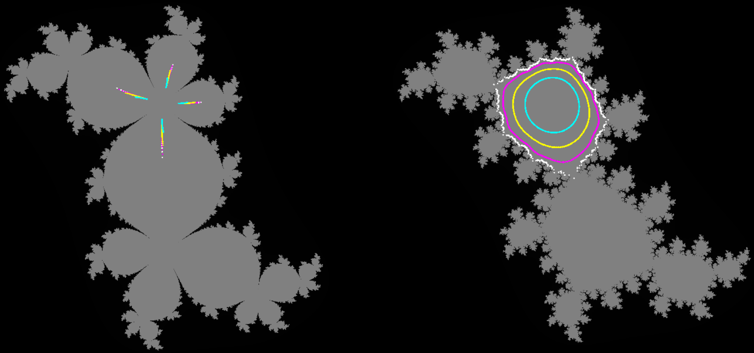

Figure: A rotation domain in $\xi \approx 0.2949$ (right), and an attracting cycle of period 4 in $\xi = 0.25$ (left).

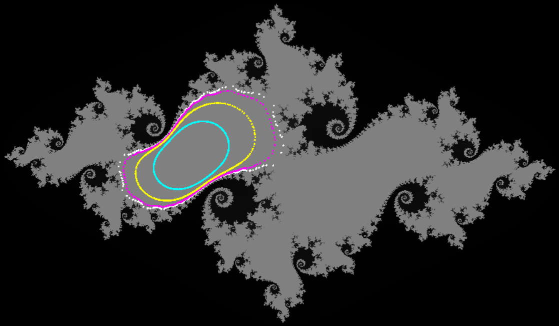

Figure: $\xi \approx 0.4892$. A rotation domain being ‘squeezed’.

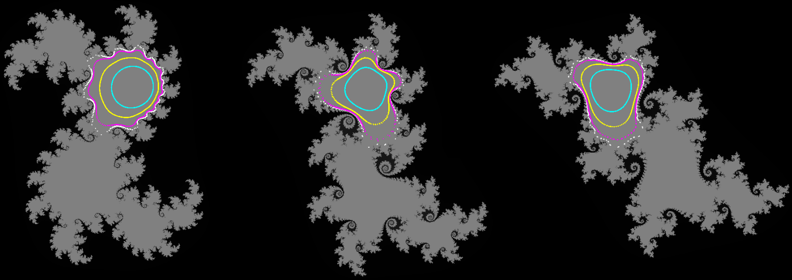

Figure: Left to right: $\xi \approx 0.1543, 0.2475, 0.3408$. Shape of rotation domain suggestive of nearby rational numbers: $2/13, 1/4, 1/3$.

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