Fractional Derivatives as Binomial Limits

Research Question:
Can the limit form of the higher-order derivative be extended to fractional orders?

(Mathematics)

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Contents

§1 - INTRODUCTION

The higher-order derivative operator can be denoted as D^n in Euler notation (or equivalently $\frac{d^n}{dx^n}$ in Leibniz notation), where $D = \frac{d}{dx}$ is the first-order derivative operator and n is any integer (n can even be negative, resulting in an integral over a $-n$ dimensional region, or 0, resulting in the identity operator).

The "fractional derivative" is a class of operators that generalises D^n to non-integer values of n, including rational, real and even complex values. The motivating property of taking rational powers of the derivative operator, in particular, is that $(D^{\frac{p}{q}})^q = D^p$ (colloquially, $D^{\frac{p}{q}}$ is the "q^{th} root" of D^p operator) for integers p, q. Such a construction is not necessarily unique, because there are often multiple "q^{th} roots" of such an operator (for an analogy, given that $F^2 \sin(x) = -\sin(x)$ for an unknown operator F, then F, that is, the square root of F^2, could be the derivative operator, or multiplication by i, or multiplication by $-i$, among other things).

In this paper, I deduce a special fractional derivative operator in limit form, similar to the standard definition of the derivative $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

My motivation for this comes from studying the proof of the wave equation from physics[1], where the limit ($h \to 0$) of $\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$ appears directly from studying the force on a small element of the string, which happens to be the same as $D^2 f$, which gives us the wave

[1] Stein, Elias M, and Rami Shakrachi. 2003. "Fourier Analysis: An Introduction." In *Princeton lectures in Analysis*, by Elias M Stein and Rami Shakrachi, 6-7. Princeton: Princeton University Press.

equation. It is possible that a similar limit-based expression for a fractional derivative may appear elsewhere in mathematics or physics, but has not yet been recognized as such.

After finding the general form of a higher-order derivatives for integer orders, I propose a generalised form for non-integer orders (in fact, there are two separate but similar generalisations that depend on the direction that the limit is taken in). The generalization is motivated by Newton's binomial theorem, which generalizes the binomial theorem to non-integer exponents. The general reason why this generalization works is the existence of the "$\binom{n}{k}$" binomial coefficients in the expansion, which vanishes for $k > n$ only for integer n.

Finally, I demonstrate the computation of the fractional derivatives of a few simple functions, but not before proving that our fractional derivative satisfies some basic properties one would expect of it, and defining a new, simpler notation which makes the whole job of dealing with such infinite sums a lot cleaner.

IN SHORT, while there exist fractional derivative operators in the literature, there are multiple ways to generalize an operator to a larger domain based on which defining properties of the operator one wishes to preserve. In this paper, we will make our generalization based on the **limit-based definition of the higher-order derivative** (Theorem 1). The primary novel contribution of this paper is of course the **generalization itself** (made in Definitions 1 and 2) – in fact, it turns out that two distinct generalisations are possible, and we call them the **right-handed and left-handed fractional derivatives** which we will denote as $D_{\rightarrow}^{T}$ and $D_{\leftarrow}^{T}$ respectively. We note a similarity between our generalization and Newton's Binomial theorem, prompting the construction of a **new notation** to represent differentials, which helps prove several **properties** of the generalization. We end by computing **special cases/examples** of our fractional derivative operator.

§2 - FOR INTEGER POWERS

The n^{th} derivative is recursively defined as $D^n f = D^{n-1}Df$, where n is an integer, and I will use Euler's notation (of representing the derivative operator with "D") and Lagrange's notation ($f''(x)$, $f^{(n)}(x)$, …) throughout this paper until we develop a new notation in Section 5. Negative values of n yield multiple indefinite (the indefiniteness of the integral can be understood to be a result of the Hilbert-space matrix for the derivative being non-invertible) integrals (single, double, triple and so on), and $n = 0$ is the identity operator.

So let's start with the second derivative.

$$D^2 f(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

Now we replace the $f'(x)$, $f'(x + h)$ with their own limit forms (the limit form for $f'(\text{x})$ evaluated at x and $x+h$ respectively):

$$\begin{aligned} D^2 f(x) &= \lim_{h\to 0} \frac{\dfrac{f(x+2h) - f(x+h)}{h} - \dfrac{f(x+h) - f(x)}{h}}{h} \\ &= \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \end{aligned}$$

Now let's do the same thing with D^3, starting with the expression above.

$$\begin{aligned} D^3 f(x) &= \lim_{h\to 0} \frac{f'(x+2h) - 2f'(x+h) + f'(x)}{h^2} \\ &= \lim_{h\to 0} \frac{\dfrac{f(x+3h) - f(x+2h)}{h} - 2\dfrac{f(x+2h) - f(x+h)}{h} + \dfrac{f(x+h) - f(x)}{h}}{h^2} \\ &= \lim_{h\to 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3} \end{aligned}$$

The coefficients of the terms in the numerator show a suspicious resemblance to the rows of Pascal's triangle/to binomial coefficients $\binom{3}{k}$ for $0 \le k \le 3$, with alternating signs. Let's expand another one in this form.

$$\begin{aligned} D^4 f(x) &= \lim_{h\to 0} \frac{f'(x+3h) - 3f'(x+2h) + 3f'(x+h) - f'(x)}{h^3} \\ &= \lim_{h\to 0} \frac{1}{h^3}\left[\frac{f(x+4h)-f(x+3h)}{h} - 3\frac{f(x+3h)-f(x+2h)}{h}\right. \\ &\qquad \left. +3\frac{f(x+2h)-f(x+h)}{h} - \frac{f(x+h)-f(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) + f(x)}{h^4} \end{aligned}$$

It is clear by now why this is the case – at each stage, the coefficient on each term comes from the sum of the coefficients on two consecutive terms in the previous limit. Since the coefficients on the terms of the last line are from Pascal's triangle, the coefficients on the terms of this line is a sequence of terms formed by adding consecutive elements of that line of Pascal's triangle – i.e. the next line of Pascal's triangle.

This rule can be illustrated via Figure 1, a diagram of Pascal's triangle taken as the array of coefficients on the terms appearing in the limit forms of the higher-order derivatives. A *divergence* of arrows from an entry in Pascal's triangle indicates a minus sign between the labels on the diverging arrows and a division by h (e.g. diverging arrows with labels $1f(x+h)$ and $1f(x)$ indicates $\frac{1f(x+h)-1f(x)}{h}$). A *convergence* of arrows towards an entry indicates the addition of the labels on the arrows to form the values of the entry itself (e.g. converging arrows with labels $1f(x+2h)$ and $2f(x+2h)$ result in the entry taking a value of $3f(x+2h)$).

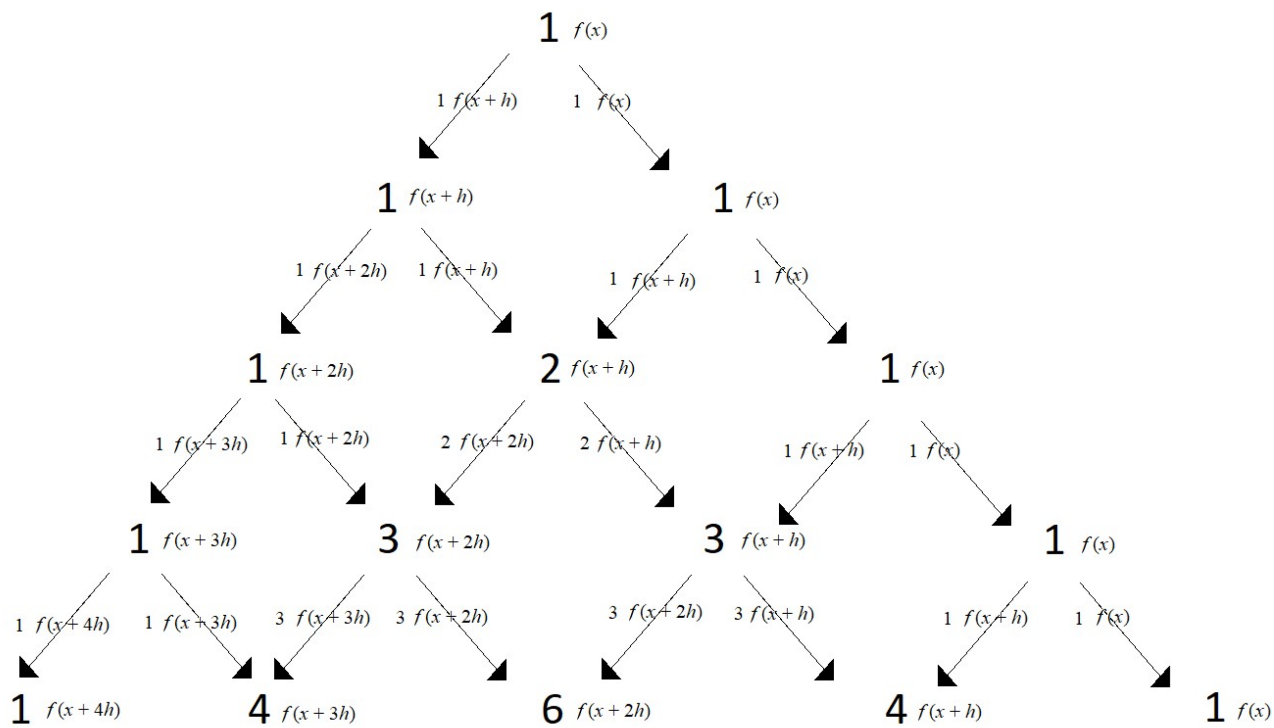

Figure 1: Pascal's triangle as array of coefficients to terms of limit numerator

We now formalize this intuition into a proof – the nature of the intuition strongly suggests an **inductive proof**. To formulate the statement of our theorem, we rewrite our results so far in the following notation:

$$D^0 f(x) = \lim_{h\to 0}\frac{1}{h^0}\sum_{k=0}^{0}(-1)^k\binom{0}{k}f\left(x+(0-k)h\right)$$
$$D^1 f(x) = \lim_{h\to 0}\frac{1}{h^1}\sum_{k=0}^{1}(-1)^k\binom{1}{k}f\left(x+(1-k)h\right)$$
$$D^2 f(x) = \lim_{h\to 0}\frac{1}{h^2}\sum_{k=0}^{2}(-1)^k\binom{2}{k}f\left(x+(2-k)h\right)$$
$$D^3 f(x) = \lim_{h\to 0}\frac{1}{h^3}\sum_{k=0}^{3}(-1)^k\binom{3}{k}f\left(x+(3-k)h\right)$$
$$D^4 f(x) = \lim_{h\to 0}\frac{1}{h^4}\sum_{k=0}^{4}(-1)^k\binom{4}{k}f\left(x+(4-k)h\right)$$

Which we may generalize to the following conjecture:

THEOREM 1: $D^N f(x) = \lim_{h\to 0} \frac{1}{h^N} \sum_{k=0}^{N} (-1)^k \binom{N}{k} f\left(x + (N-k)h\right)$

Proof

First, observe that when N = 1, the right-hand-side reduces to $\lim_{h\to 0} \frac{1}{h}\left[\binom{1}{0} f(x+h) - \binom{1}{1} f(x)\right]$

$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$*, which is equal to (by definition),* $Df(x)$*. Hence the statement is true for the base case N = 1. Now suppose that the proposition is true for some N = M. Then*

$$\begin{aligned}
D^{M+1} f(x) &= D^M Df(x) \\
&= \frac{1}{h^M}\left[\sum_{k=0}^{M} (-1)^k \binom{M}{k} f'\left(x + (M-k)h\right)\right] \\
&= \frac{1}{h^M}\left[\sum_{k=0}^{M} (-1)^k \binom{M}{k} \frac{f\left(x + (M-k+1)h\right) - f\left(x + (M-k)h\right)}{h}\right] \\
&= \frac{1}{h^{M+1}}\left[\sum_{k=0}^{M} (-1)^k \binom{M}{k} \left[f\left(x + \left((M+1) - k\right)h\right) - f\left(x + (M-k)h\right)\right]\right]
\end{aligned}$$

Our desired end result contains all the f (x + bh) terms grouped together (e.g. all the coefficients on say for instance, $f\left(x+3h\right)$ *terms, would be added up, rather than have multiple multiples of* $f\left(x+3h\right)$*, so let's regroup the terms:*

$$\begin{aligned}
D^{M+1} f(x) &= \frac{1}{h^{M+1}}\left[\left[\sum_{k=0}^{M} (-1)^k \binom{M}{k} f\left(x + \left((M+1) - k\right)h\right)\right] - \left[\sum_{k=0}^{M} (-1)^k \binom{M}{k} f\left(x + \left((M-k\right)h\right)\right]\right] \\
&= \frac{1}{h^{M+1}}\left[\left[\sum_{k=0}^{M} (-1)^k \binom{M}{k} f\left(x + \left((M+1) - k\right)h\right)\right] - \left[\sum_{k=0}^{M} (-1)^k \binom{M}{k} f\left(x + \left((M+1) - (k+1)\right)h\right)\right]\right]
\end{aligned}$$

We further split the first summation into the k = 0 term and the rest of the terms, and replace the index of the second summation $k \leftarrow k+1$ *to help us group terms together.*

$$D^{M+1}f(x)=\frac{1}{h^{M+1}}\left[\left[(-1)^0\binom{M}{0}f\Big(x+\big((M+1)-0\big)h\Big)\right]+\left[\sum_{k=1}^{M}(-1)^k\binom{M}{k}f\Big(x+\big((M+1)-k\big)h\Big)\right]\right.$$

$$\left.-\left[\sum_{k=1}^{M}(-1)^{k-1}\binom{M}{k-1}f\Big(x+\big((M+1)-k\big)h\Big)\right]\right]$$

$$=\frac{1}{h^{M+1}}\left[f\Big(x+\big(M+1\big)h\Big)+\left[\sum_{k=1}^{M}(-1)^k\binom{M}{k}f\Big(x+\big((M+1)-k\big)h\Big)\right]\right.$$

$$\left.+\left[\sum_{k=1}^{M}(-1)^k\binom{M}{k-1}f\Big(x+\big((M+1)-k\big)h\Big)\right]\right]$$

$$=\frac{1}{h^{M+1}}\left[f\Big(x+\big(M+1\big)h\Big)+\left[\sum_{k=1}^{M}(-1)^k\left(\binom{M}{k}+\binom{M}{k-1}\right)f\Big(x+\big((M+1)-k\big)h\Big)\right]\right]$$

$\binom{M}{k}$ and $\binom{M}{k-1}$ are just the k^{th} and $(k + 1)^{th}$ terms of the M^{th} row of Pascal's triangle. Their sum is equal to the element of Pascal's triangle in the row below them, placed horizontally in between, i.e. $\binom{M+1}{k}$. So we rewrite this as:

$$D^{M+1}f(x)=\frac{1}{h^{M+1}}\left[f\Big(x+\big(M+1\big)h\Big)+\left[\sum_{k=1}^{M}(-1)^k\binom{M+1}{k}f\Big(x+\big((M+1)-k\big)h\Big)\right]\right]$$

$$=\frac{1}{h^{M+1}}\left[(-1)^0\binom{M+1}{0}f\Big(x+\big((M+1)-0\big)h\Big)+\left[\sum_{k=1}^{M}(-1)^k\binom{M+1}{k}f\Big(x+\big((M+1)-k\big)h\Big)\right]\right]$$

$$=\frac{1}{h^{M+1}}\left[\sum_{k=0}^{M}(-1)^k\binom{M+1}{k}f\Big(x+\big((M+1)-k\big)h\Big)\right]$$

Which is our desired result for N = M + 1. Since our proposition is true for N = 1 and true for any M + 1 if it is true for M, it must therefore be true for N = 2, 3, 4… and by induction, for all N.

It is interesting to observe that the sum of the coefficients $\sum_{k=0}^{M}(-1)^k\binom{M}{k}$ is equal to $\sum_{k=0}^{M}\binom{M}{k}(-1)^k 1^{M-k}$, which is the binomial expansion for $(1 - 1)^k$ and therefore equal to zero. Therefore directly setting $h = 0$ will always directly yield 0/0, an undefined form, and evaluating the limit is necessary.

Examples and properties

We will verify Theorem 1 with a few examples based on some higher-order derivatives we already know.

1. Third derivative of $f(x) = x^2$ at $x = 4$ (value arbitrarily chosen to simplify calculation).

   The limit we're interested in is $\lim_{h\to 0}\frac{(x+3h)^2-3(x+2h)^2+3(x+h)^2-x^2}{h^3}$. Expanding this out, we get $\lim_{h\to 0}\frac{x^2+6hx+9h^2-\left(3x^2+12hx+12h^2\right)+\left(3x^2+6hx+3h^2\right)-x^2}{h^3}$, and the numerator simplifies to 0 regardless of x. Thus the limit is 0, including when $x = 4$, which confirms our standard knowledge about the third derivative of a quadratic polynomial.

2. Second derivative of $f(x) = \sin(x)$ at $x = 0.25\pi$.

   The limit we're interested in is $\lim_{h\to 0}\frac{\sin(x+2h)-2\sin(x+h)+\sin x}{h^2}$. We may use L' Hopital's rule to evaluate the limit (since we already know what the derivatives of sin(x) are, and are just verifying that our limit works) – we find that after differentiating the numerator and denominator with respect to h, the limit becomes $\lim_{h\to 0}\frac{2\cos(x+2h)-2\cos(x+h)}{2h}=-\sin x$, which produces $-1/\sqrt{2}$ at $x = 0.25\pi$.

3. Linearity of D^N, i.e. $D^N[af(x)] = aD^N f(x)$ and $D^N[f(x) + g(x)] = D^N f(x) + D^N g(x)$.

(i) $$D^N\left[af(x)\right] = \lim_{h\to 0}\frac{1}{h^N}\sum_{k=0}^{N}(-1)^k\binom{N}{k}af\left(x+(N-k)h\right)$$
$$= a\lim_{h\to 0}\frac{1}{h^N}\sum_{k=0}^{N}(-1)^k\binom{N}{k}f\left(x+(N-k)h\right)$$
$$= aD^N f(x)$$

(ii) $$D^N\left[f(x)+g(x)\right] = \lim_{h\to 0}\frac{1}{h^N}\sum_{k=0}^{N}(-1)^k\binom{N}{k}\left(f\left(x+(N-k)h\right)+g\left(x+(N-k)h\right)\right)$$
$$= \lim_{h\to 0}\frac{1}{h^N}\left[\sum_{k=0}^{N}(-1)^k\binom{N}{k}f\left(x+(N-k)h\right)\right]$$
$$+\lim_{h\to 0}\frac{1}{h^N}\left[\sum_{k=0}^{N}(-1)^k\binom{N}{k}g\left(x+(N-k)h\right)\right]$$
$$= D^N f(x) + D^N g(x)$$

§3 - THE GENERALISATION

Analogy with Newton's binomial theorem

Understanding of the motivation for this generalization warrants a brief discussion of Newton's binomial theorem, which made a similar generalization of the standard binomial theorem. The analogy is useful, because the sum in the numerator of the limit closely resembles the general binomial expansion of the expression $(t-1)^N$ except t^{N-k} is replaced with $f(x+(N-k)h)$.

Newton's binomial theorem generalizes the standard binomial theorem $(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^{N-k} b^k$ to non-integer (rational, real or potentially complex) powers by **replacing the summation with an infinite sum:**

$$(a+b)^N = \sum_{k=0}^{\infty} \binom{N}{k} a^{N-k} b^k$$

Where

$$\binom{N}{k} = \frac{N(N-1)\ldots(N-k+1)}{k!}$$

Is the **generalised binomial coefficient** defined for potentially non-integer N but integer k[2].

Replacing the summation with an infinite summation does not make a difference for integer N as $\binom{N}{k}$ is zero for $k > N$. However, the resulting expression agrees with the Taylor expansion when N is not an integer.

[2] Weisstein, Eric. 2000. *Binomial Theorem.* May 12. Accessed April 11, 2017. http://mathworld.wolfram.com/BinomialTheorem.html.

Once again, for integer N this reduces to the standard binomial coefficient. It is important to note that for $\binom{N}{k}$ to be equal to zero, at least one of the terms of the product in the numerator must be 0. However, for non-integer N, all of 0, 1, … $k-1$ are integers while N is not, therefore N cannot be equal to any of them, and thus none of the terms is zero. On the other hand for non-negative integer N, $\binom{N}{k}=0$ would require that N is equal to one of 0, 1, … $k-1$, which is satisfied if and only if $N \le k-1 \Leftrightarrow k > N$ and $N \ge 0$.

Examples

1. $\binom{3.5}{2} = \frac{3.5\times 2.5}{2!} = 4.375$
2. $\binom{4.5}{8} = \frac{4.5\times 3.5\times 2.5\times 1.5\times 0.5\times(-0.5)\times(-1.5)\times(-2.5)}{8!} = -\frac{14175}{2^8\times 8!} \approx 0.0014$
3. Terms 0 to 5 for the 2.5th row of Pascal's triangle –

$$\binom{2.5}{0}=1;\ \binom{2.5}{1}=2.5;\ \binom{2.5}{2}=1.875;\ \binom{2.5}{3}=0.3125;\ \binom{2.5}{4}=-0.0391;\ \binom{2.5}{5}=0.0117$$

Proposing the fractional derivative

We make the same generalization to generalize the limit expression to fractional powers of the derivative operator by replacing the summation in Theorem 1 (p. 7) with an infinite sum.

DEFINITION 1: For any real number T, we define the **right-handed fractional derivative** $D^T_{\rightarrow} f(x)$ as follows:

$$D^T_{\rightarrow} f(x) = \lim_{h\to 0}\frac{1}{h^T}\sum_{k=0}^{\infty}\binom{T}{k}(-1)^k f\left(x+\left(T-k\right)h\right)$$

The reason I call this the *right-handed* derivative is that it turns out that this is not the only possible generalization of the limit expression to fractional powers of the derivative operator. Observe that this expression starts with the term $f(x + Th)$ and counts down as $f(x + (T - 1)h)$, $f(x + (T - 2)h)$, $f(x + (T - 3)h)$… For values of T that are not positive integers, this sequence does not include the term $f(x)$, since there is no non-negative integer k such that $T - k = 0$.

However, this is purely a result of the ordering we defined in Theorem 1 (p. 7) while making the generalization – we started the series with $f(x + Nh)$ and counted down. In other words, we ordered the integer-order derivative as:

$$D^N f(x) = \lim_{h \to 0} \frac{1}{h^N}\left[f\left(x+Nh\right) - Nf\left(x+\left(N-1\right)h\right) + \frac{N(N-1)}{2} f\left(x+\left(N-2\right)h\right) - \ldots + \left(-1\right)^N \frac{N(N-1)}{2} f(x+2h) - \left(-1\right)^N Nf(x+h) + \left(-1\right)^N f(x) \right]$$

However, we could also have re-ordered the terms so that the expansion starts with $f(x)$ and continues to $f(x + Nh)$.

$$D^N f(x) = \lim_{h \to 0} \frac{1}{h^N}\left[\left(-1\right)^N f(x) - \left(-1\right)^N Nf(x+h) - \ldots + \frac{N(N-1)}{2} f\left(x+\left(N-2\right)h\right) - Nf\left(x+\left(N-1\right)h\right) + f\left(x+Nh\right) \right]$$

Which can be written in summation notation as:

$$D^N f(x) = \lim_{h \to 0} \frac{1}{h^N} \sum_{k=0}^{N} \binom{N}{k} \left(-1\right)^{N-k} f\left(x+kh\right)$$

Or:

$$D^N f(x) = \lim_{h \to 0} \frac{\left(-1\right)^N}{h^N} \sum_{k=0}^{N} \binom{N}{k} \left(-1\right)^{k} f\left(x+kh\right)$$

This is equivalent to the result in Theorem 1 (p. 7), as shown above (since the new summation was obtained simply by re-ordering the terms in Theorem 1). However, this may not be the case with our generalization in Definition 1 (p. 12). This is a result of the fact that the identity $\binom{N}{k} = \binom{N}{N-k}$ does not generalize to non-integer N – in fact, if N is not an integer and either one of the two is defined, the other one must be undefined as lower number will not be an integer (it's only the upper number that need not be an integer in the generalised binomial coefficient). For example if N is 3.5 and k is 2, then $\binom{N}{k}$ is defined. However, $N-k=1.5$ and $\binom{N}{N-k}$ is no longer defined. Since the identity does not generalize, the binomial coefficients on the terms will be different in the two alternative expansions (the expansion in Definition 1 and the following expansion in Definition 2).

We make the generalization in the same way as before, replacing the sum with an infinite sum.

DEFINITION 2: For any real number T, we define the **left-handed fractional derivative** $D^{T}_{\leftarrow} f(x)$ as follows:

$$D^{T}_{\leftarrow} f(x) = \lim_{h\to 0} \frac{(-1)^{T}}{h^{T}} \sum_{k=0}^{\infty} \binom{T}{k} (-1)^{k} f\left(x+kh\right)$$

You may notice that the left-handed derivative seems to be quite similar to the right-handed one except for a few sign changes. Indeed, if we make the substitution $h \to -h$, we see that:

$$D^{T}_{\leftarrow} f(x) = \lim_{h\to 0} \frac{1}{h^{T}} \sum_{k=0}^{\infty} \binom{T}{k} (-1)^{k} f\left(x-kh\right)$$

Note that $x+Th=x$ when h approaches zero, therefore this is equivalent to

$$D_{\leftarrow}^{T} f(x) = \lim_{h \to 0} \frac{1}{h^T} \sum_{k=0}^{\infty} \binom{T}{k} (-1)^k f\left(x + (T-k)h\right)$$

However, the two fractional derivatives are not necessarily equivalent, because the substitution $h \to -h$ means that we're taking the limit from the opposite direction. Therefore, it is fair to say that the two derivatives are equivalent when and only when the limit exists. In this circumstance, we will simply denote our fractional derivative as $D^T f(x)$.

Recall our earlier comment (beginning of p. 9) on the fact that the limit in Theorem 1 (p. 7) needs to be computed because directly setting $h = 0$ results in an indeterminant form. It is important to verify that this is the case with our generalization (the left and right handed fractional derivatives) as well if it is to be useful. In other words, we need to prove that $\lim_{h \to 0} \sum_{k=0}^{\infty} \binom{T}{k} (-1)^k f\left(x + \left(T-k\right)h\right)$ converges to 0.

§4 - PROOF OF CONVERGENCE

THEOREM 2: The limit $\lim_{h\to 0}\frac{1}{h^T}\sum_{k=0}^{\infty}(-1)^k\binom{T}{k}f\left(x+(T-k)h\right)$, when directly evaluated (i.e. by setting h = 0), yields an indeterminate form when $f(x)$ is not the zero function.

<u>*Proof*</u>

The point of the theorem is that the limit yields a 0/0 form, so it is not a constant value for all functions, and doesn't yield something like 1/0 which never makes sense.

In order to prove this, it is sufficient to show that $\sum_{k=0}^{\infty}\binom{T}{k}(-1)^k f\left(x+(T-k)h\right)$ equals $\sum_{k=0}^{\infty}(-1)^k\binom{T}{k}f\left(x+(T-k)\cdot 0\right)$ *is 0 for positive T and divergent for negative T. Since f(x) is not the zero function, we are in effect evaluating the sum:*

$$\sum_{k=0}^{\infty}(-1)^k\binom{T}{k}$$

The easiest way to calculate this sum is to go back to Newton's binomial theorem, noting that

$$\sum_{k=0}^{\infty}(-1)^k\binom{T}{k}=\sum_{k=0}^{\infty}(-1)^k\binom{T}{k}1^{T-k}1^k=(1-1)^T=0^T$$

Indeed, 0^T *is 0 whenever T > 0 and undefined otherwise, as required.*

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§5 - PROPERTIES

Some elementary results regarding right and left-handed fractional derivatives are listed below:

- After terms $k = 0 \ldots \lfloor T \rfloor + 1$ (where $\lfloor T \rfloor$ is the greatest integer lower than *T*), the rest of the terms all have the same sign because while the sign of $(-1)^k$ keeps flipping, so does the sign of $\binom{T}{k}$, because a new negative number is multiplied each time.
- The fractional derivative is linear, i.e. the fractional derivative of a sum of two functions is the sum of the fractional derivatives of each function, and $D^T kf(x) = kD^T f(x)$. This follows from the linearity of the summation operator itself.

Before we show a few more properties of these operators, we first introduce the following notation:

- Consider some operator (specifically a functional) H_{x+k} that maps the function *f* to the scalar value $f(x+k)$. We will write this operator as φ^k, suppressing the *x* (the value of *x* should be inputted separately), i.e. $\varphi^k f\big|_x = f(x+k)$ for all functions *f* and elements *x* and *k* in the domain of *f*. As an example, $\varphi^\pi \cos\big|_{2\pi} = -1$.
- Here φ is a special mathematical object (*not* a real or complex number or any such standard thing) that satisfies the properties $\varphi^a \varphi^b = \varphi^{a+b}$ and $\left(\varphi^a\right)^m = \varphi^{am}$, justifying the use of exponent notation (as the defining properties of exponents are satisfied).
- We say that the differential operator (*not* differentiation operator $D = \frac{d}{dx}$) $d = \varphi^h - \varphi^0$, where *d* is the same *d* as in $\frac{d}{dx}$ or $\frac{d^2}{dx^2}$ and *h* is an infinitesimal (i.e. it approaches 0).

Where the last definition of the differential operator is reasonable because $df = f(x+h) - f(x)$ (from the definition of the derivative $\frac{df}{dx}$), which is the same as $\varphi^h - \varphi^0\big|_x$.

The motivation for using this notation is that it makes the entirety of Theorem 1, as well as our generalisations in Definition 1 (p. 12) and Definition 2 (p. 14) much more natural, as direct results of the Binomial theorem by binomially expanding out the numerator of $\frac{d^2}{dx^2}$ after substituting $d = \varphi^h - \varphi^0$ (yes, it causes Leibniz notation for repeated differentiation to make a lot more sense, considering that *h* is simply alternative notation for *dx*). For example,

$$\begin{aligned} d^3 &= \left(\varphi^h\right)^3 - 3\left(\varphi^h\right)^2\left(\varphi^0\right) + 3\left(\varphi^h\right)\left(\varphi^0\right)^2 - \left(\varphi^0\right)^3 \\ &= \varphi^{3h} - 3\varphi^{2h} + 3\varphi^h - \varphi^0 \\ &\Rightarrow \frac{d^3}{dx^3} = \frac{\varphi^{3h} - 3\varphi^{2h} + 3\varphi^h - \varphi^0}{h^3} \end{aligned}$$

Which is the limit expansion for the third derivative (a special case of Theorem 1) expressed in this new notation. Indeed, using the definition of φ and the fact that $h \to 0$, this can be written in standard notation as:

$$\frac{d^3}{dx^3} = \lim_{h \to 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3}$$

This notation makes Definition 1 (p. 12) and Definition 2 (p. 14) simply the Newtonian binomial expansion of $d^T = \left(\varphi^h - \varphi^0\right)^T$ for non-integer *T*, and Theorem 1 a case of the integer-power binomial theorem, i.e. simply the binomial expansion of $d^N = \left(\varphi^h - \varphi^0\right)^N$ for positive integer *N*. This makes it easier to find the properties we're looking for – the properties follow:

1. $D^R D^S = \left(\left(\varphi^h - \varphi^0\right)/h\right)^R \left(\left(\varphi^h - \varphi^0\right)/h\right)^S$

$$= \left(\left(\varphi^h - \varphi^0\right)/h\right)^{R+S}$$
$$= D^{R+S}$$

2. $\left(D^R\right)^S = \left(\left(\left(\varphi^h - \varphi^0\right)/h\right)^R\right)^S = \left(\left(\varphi^h - \varphi^0\right)/h\right)^{RS} = D^{RS}$

3. $\left(D^M\right)^{\frac{1}{M}} = D^{M \cdot \frac{1}{M}} = D$

4. $\left(D^{\frac{P}{Q}}\right)^Q = D^P$

5. $D^T D^{-T} = D^{T-T} = I$ where I is the identity operator.

These properties are much more difficult to show in the fully-expanded limit form. The fourth property, in particular, is the condition for a valid fractional derivative[3], and the fact that our generalization satisfies it is proof that it is a fractional derivative at all.

[3] Bologna, Mauro. 2014. "Short Introduction to Fractional Calculus." *Universidad de Tarapaca*. April 7. Accessed May 5, 2017. http://uta.cl/charlas/volumen19/Indice/MAUROrevision.pdf.

§6 - COMPUTING SPECIAL CASES

Extracting the indefinite integral from the fractional derivative

As an example, let's try to simplify $D_{\rightarrow}^{-1} f(x)$ by setting $T = -1$ in Definition 1 (p. 12).

$$\begin{aligned}
D_{\rightarrow}^{-1} f(x) &= \lim_{h\to 0} \sum_{k=0}^{\infty} \binom{-1}{k} (-1)^k f\big(x-(1+k)h\big)h \\
&= \lim_{h\to 0} \sum_{k=0}^{\infty} \frac{(-1)(-2)...(-k)}{k!} (-1)^k f\big(x-(1+k)h\big)h \\
&= \lim_{h\to 0} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{k!} (-1)^k f\big(x-(1+k)h\big)h \\
&= \lim_{h\to 0} \sum_{k=0}^{\infty} f\big(x-(1+k)h\big)h \\
&= \lim_{h\to 0} \big[f(x-h)h + f(x-2h)h + ...\big]
\end{aligned}$$

Observe that this is just the Riemann sum for $\int_a^x f(t)\, dt$, where $a < x$. Since a is unknown, this is simply the indefinite integral $\int f(x)\, dx$. Analogously, for the left-handed derivative:

$$\begin{aligned}
D_{\leftarrow}^{-1} f(x) &= \lim_{h\to 0} \sum_{k=0}^{\infty} \binom{-1}{k} (-1)^{-1-k} f\big(x+kh\big)h \\
&= -\lim_{h\to 0} \sum_{k=0}^{\infty} \binom{-1}{k} (-1)^k f\big(x+kh\big)h \\
&= -\lim_{h\to 0} \sum_{k=0}^{\infty} \frac{(-1)(-2)...(-k)}{k!} (-1)^k f\big(x+kh\big)h \\
&= -\lim_{h\to 0} \sum_{k=0}^{\infty} f\big(x+kh\big)h \\
&= -\lim_{h\to 0} \big[f(x)h + f(x+h)h + f(x+2h)h + ...\big] \\
&= -\int_x^a f(t)\, dt \\
&= \int_a^x f(t)\, dt \\
&= \int f(x)\, dx
\end{aligned}$$

Fractional derivative of a constant

Here, we're calculating the limit $\lim_{h\to 0}\frac{1}{h^T}\sum_{k=0}^{\infty}(-1)^k\binom{T}{k}C$ for some constant *C*. Computing the summation as we did in Theorem 2, this is equivalent to $\lim_{h\to 0}\frac{1}{h^T}0^T C$. When $T > 0$, i.e. when we're computing a derivative, not an integral, the limit evaluates to zero.

Half-derivative of a linear function.

Consider the function $f(x) = x$ and substitute it into the formula for the right-handed fractional derivative of order $\frac{1}{2}$ – note that the term $x+\left(\frac{1}{2}-k\right)h$ in the summand can be replaced with $x-kh$ since $x+\frac{1}{2}h = x$ when *h* approaches zero.

$$D^{\frac{1}{2}}(x)=\lim_{h\to 0}\frac{1}{h^{\frac{1}{2}}}\sum_{k=0}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}(x-kh)=\lim_{h\to 0}\frac{1}{\sqrt{h}}\left(x\sum_{k=0}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}-h\sum_{k=0}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}k\right)$$

To evaluate the sum $\sum_{k=0}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}$, we refer once again to Newton's Binomial Theorem:

$$\sum_{k=0}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}=\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-1)^k\,1^{\frac{1}{2}-k}=(1-1)^{\frac{1}{2}}=0$$

Therefore

$$D^{\frac{1}{2}}(x)=-\lim_{h\to 0}\sqrt{h}\sum_{k=0}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}k$$

Which we may further simplify as follows:

$$D^{1/2}(x) = -\lim_{h\to 0}\sqrt{h}\sum_{k=0}^{\infty}\left(-1\right)^k \begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix} k$$

$$= -\lim_{h\to 0}\sqrt{h}\sum_{k=0}^{\infty}\left(-1\right)^k \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(\frac{1}{2}-\left(k-1\right)\right)}{k!} k$$

$$= -\sqrt{h}\sum_{k=0}^{\infty}\left(-1\right)^k \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{2k-3}{2}\right)}{k!} k$$

$$= -\sqrt{h}\sum_{k=0}^{\infty}(-1)^k(-1)^{k-1}\frac{\frac{1}{2}\frac{1}{2}\frac{3}{2}\ldots\frac{2k-3}{2}}{k!} k$$

$$= \sqrt{h}\sum_{k=0}^{\infty}\frac{\frac{1}{2}\frac{1}{2}\frac{3}{2}\ldots\frac{2k-3}{2}}{k!} k$$

$$= \sqrt{h}\sum_{k=1}^{\infty}\frac{\frac{1}{2}\frac{1}{2}\frac{3}{2}\ldots\frac{2k-3}{2}}{k!} k$$

Where the last step is acceptable because when the index $k = 0$, the summand clearly evaluates to $\frac{\frac{1}{2}\frac{1}{2}\frac{3}{2}\ldots\frac{2k-3}{2}}{k!} k = 0$. We now try to convert this summation into an index of j, ranging from 0 to infinity, where $j = k - 1$.

$$D^{1/2}(x) = \frac{\sqrt{h}}{2}\sum_{j=0}^{\infty}\frac{\frac{1}{2}\frac{3}{2}\ldots\frac{2j-1}{2}}{j!}$$

$$= \frac{\sqrt{h}}{2}\sum_{j=0}^{\infty}\frac{1\times 3\times\ldots(2j-1)}{2^j j!}$$

$$= \frac{\sqrt{h}}{2}\sum_{j=0}^{\infty}\frac{(2j)!}{(2\times 4\times\ldots 2j)2^j j!}$$

$$= \frac{\sqrt{h}}{2}\sum_{j=0}^{\infty}\frac{(2j)!}{(1\times 2\times\ldots j)2^j 2^j j!}$$

$$= \frac{\sqrt{h}}{2} \sum_{j=0}^{\infty} \binom{2j}{j} \left(\frac{1}{4}\right)^j$$

Taking a few partial sums indicates that this sum is divergent, and Wolfram Alpha indicates that the solution involves the usage of the comparison test. This divergence can be proven as follows:

We try to compare the term $\binom{2j}{j}\left(\frac{1}{4}\right)^j$ with $\frac{1}{j+1}$ (j + 1, because $\frac{1}{j}$ is undefined for $j = 0$ while our term is not). If we can prove that $\binom{2j}{j}\left(\frac{1}{4}\right)^j \geq \frac{1}{j+1}$ for all non-negative j, we will have proved that the series is divergent, since the harmonic series (the sum of terms $\frac{1}{j+1}$ over the index j) is itself divergent.

The required condition is equivalent to $4^j \leq (j+1)\binom{2j}{j}$. On Pascal's triangle, $\binom{2j}{j}$ is the central term (i.e. the largest term) of the $2j^{\text{th}}$ row, $j + 1$ the number of terms, and 4^j the sum of terms in the row. Then $(j+1)\binom{2j}{j}$ is simply what the sum *would have been* if every term were as large as the central term. Naturally, this is greater than the actual sum, and the inequality holds.

Thus, the sum diverges, and the half-derivative of $f(x) = x$ does not exist.

Fractional derivative of exponential and trigonometric functions

Our failure to calculate the simple half-derivative above – and in fact discovering that the function x does not have a half-derivative – makes us worry that the fractional derivative does not generally exist for most "elementary" functions. We therefore try once more with another important function, the exponential function e^{ax}. Here,

$$D^T e^{ax} = \lim_{h\to 0} \frac{1}{h^T} \sum_{k=0}^{\infty} \binom{T}{k} (-1)^k e^{a(x-kh)}$$
$$= \lim_{h\to 0} \frac{e^{ax}}{h^T} \sum_{k=0}^{\infty} \binom{T}{k} \left(-e^{-ah}\right)^k 1^{T-k}$$

Once again, we may apply Newton's generalised binomial theorem as follows:

$$\begin{aligned}
D^T e^{ax} &= \lim_{h\to 0} \frac{e^{ax}}{h^T} \sum_{k=0}^{\infty} \binom{T}{k} \left(-e^{-ah}\right)^k 1^{T-k} \\
&= \lim_{h\to 0} \frac{e^{ax}}{h^T} \left(1-e^{-ah}\right)^T \\
&= e^{ax} \lim_{h\to 0} \left(\frac{1-e^{-ah}}{h}\right)^T \\
&= e^{ax} \lim_{h\to 0} \left(\frac{ae^{-ah}}{1}\right)^T \qquad \text{(L' Hospital's rule)} \\
&= a^T e^{ax}
\end{aligned}$$

Which proves that the corresponding result for integer-order derivatives generalizes exactly to fractional derivatives.

One can now immediately calculate the fractional derivatives of cos(x) and sin(x) with this and the linearity of the fractional derivative we know from the section "Properties":

$$D^T \cos x = D^T \frac{e^{ix}+e^{-ix}}{2} = \frac{i^T e^{ix} + (-i)^T e^{-ix}}{2} = \frac{(-1)^T e^{ix} + e^{-ix}}{2i^T}$$
$$D^T \sin x = D^T \frac{e^{ix}-e^{-ix}}{2i} = \frac{i^T e^{ix} - (-i)^T e^{-ix}}{2i} = \frac{(-1)^{T-1} e^{ix} + e^{-ix}}{2i^{T-1}}$$

The expressions for both functions is intriguingly similar, a result of the fact that $\sin x$ can itself be written as $\sin x = \frac{(-1)^{-1} e^{ix} + e^{-ix}}{2i^{-1}}$ while $\cos x = \frac{e^{ix}+e^{-ix}}{2}$. Additionally, one may confirm that setting T to be an integer yields the standard sequence $\sin x, \cos x, -\sin x, -\cos x...$, confirming that the fractional derivative agrees with our existing knowledge.

§7 - CONCLUSION

We have thus produced two distinct infinite-sum expansions for a fractional derivative operator which generalizes the operator D^N to all real (or potentially even complex) values of N, yielding two separate operators – the left-handed fractional derivative and the right-handed fractional derivative. Among other properties, we were able to show that both the operators satisfy the necessary conditions for being classified as a fractional derivative operator.

The paper leaves several avenues for future research:

1. A more detailed analysis of convergence conditions for fractional derivatives – several functions whose half-derivative I tried to take seemed to be divergent at all values of x, or divergent at all values of x except 0. Yet others yielded complex solutions whose imaginary part alone converged while the real part diverged. A study of the details of the convergence criteria is thus necessary.
2. Further study is needed on the properties of the fractional derivative, such as to see if there are generalisations of the product rule, chain rule, etc. for the fractional derivative. On a related note, it might be interesting to study how the fractional derivative behaves with complex orders.
3. It might be worth studying how fractional derivatives and integrals would behave with the background of multiple variables, as I restricted my study to single-variable calculus in this paper. Similarly, the approach could be extended to other fields of analysis, such as the calculus of variations.
4. It is possible that similar to the Taylor series, which is a sum of weighed integer-order derivatives of a function, a function could also be written as a sum of weighted fractional

derivatives of itself. Such a sum would be an integral, as the space of all real numbers is the continuum.

The mathematical content of this paper has possible implications in solving fractional differential equations – a class of differential equations that are gaining prominence for modelling various phenomena[4], such as in physics, especially pertaining to fluid mechanics. Many believe that unsolved problems relating to turbulence will eventually be expressed with fractional differential equations[5].

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[4] West, Bruce J. 2014. "Colloquium: Fractional calculus view of complexity: A tutorial." *Reviews of Modern Physics* 1169-1186.
[5] Chen, Wen. 2005. "Fractional and Fractal derivatives modeling of turbulence." *ArXiv*. November 11. Accessed May 5, 2017. https://arxiv.org/ftp/nlin/papers/0511/0511066.pdf.

BIBLIOGRAPHY

1. Bologna, Mauro. 2014. "Short Introduction to Fractional Calculus." *Universidad de Tarapaca.* April 7. Accessed May 5, 2017. http://uta.cl/charlas/volumen19/Indice/MAUROrevision.pdf.
2. Chen, Wen. 2005. "Fractional and Fractal derivatives modeling of turbulence." *ArXiv.* November 11. Accessed May 5, 2017. https://arxiv.org/ftp/nlin/papers/0511/0511066.pdf.
3. Stein, Elias M, and Rami Shakrachi. 2003. "Fourier Analysis: An Introduction." In *Princeton lectures in Analysis* , by Elias M Stein and Rami Shakrachi, 6-7. Princeton: Princeton University Press.
4. Weisstein, Eric. 2000. *Binomial Theorem.* May 12. Accessed April 11, 2017. http://mathworld.wolfram.com/BinomialTheorem.html.
5. West, Bruce J. 2014. "Colloquium: Fractional calculus view of complexity: A tutorial." *Reviews of Modern Physics* 1169-1186.