An exploration of the several approaches to finding the number of terms in a multinomial expansion

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Aim: To explore numerous approaches to finding the number of terms in a multinomial expansion and demonstrate that they produce the same result.

Rationale: Rather than something in real life, the motivation for my project came from my own personal academic pursuit, where I worked with Newton's binomial theorem. I often got carried away studying the properties of the integer binomial expansion, which often correspond to some simple properties of exponentiation itself, much like how vice versa is true. This naturally lead me to studying the properties of the more general multinomial expansion.

The multinomial expansion is the algebraic expansion of the general expression $(x_1 + ... + x_p)^n$. In this paper, we suggest and prove an expression for the number of terms in this expansion using several different methods – an observed relation which we prove via mathematical induction, a recurrence relation which we solve via a generating function, and a simpler combinatoric method. We demonstrate that the resulting expression is the same despite the disconnected methods, and present a geometric interpretation of the result and its proofs.

§1. INTRODUCTION

In general, the binomial expansion takes the form:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

This expansion clearly has $n + 1$ terms in all cases.

We may consider the trinomial expansion and expand it twice as follows:

$$(x+y+z)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left(y+z\right)^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \sum_{j=0}^{k} \binom{k}{j} y^{k-j} z^j$$

Here, for any specific value of k, there is an entire summation with $k + 1$ terms. The summation at a certain value of k does not have its terms overlap with the summation of some other value of k (i.e. none of the 5 terms at $k = 4$ combine with any of the 3 terms at $k = 2$, which would otherwise reduce the number of terms), so one may write the number of terms here as $1+2+...+(n+1) = \frac{(n+1)(n+2)}{2}$ terms.

Similarly, one may expect the "tetranomial" (a polynomial with four terms) expansion to take the form of $1+3+6+...+\frac{(n+1)(n+2)}{2}$. Indeed, we may write:

$$(x+y+z+w)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left(y+z+w\right)^k$$

Since $\left(y+z+w\right)^k$ has $\frac{\left(k+1\right)\left(k+2\right)}{2}$ terms, the number of terms in the tetranomial expansion is $\sum_{k=0}^{n}\frac{\left(k+1\right)\left(k+2\right)}{2}$.

Therefore, simply by looking at the first few special cases of the multinomial expansion, we have found a pattern in how the number of terms in a p-nomial expansion relates to the number of terms in a $(p-1)$-nomial expansion – i.e. a recursion relation. We will formalize this relation, and use it as the basis for the first two approaches we will study.

§2. VIA MATHEMATICAL INDUCTION

The first approach we will employ is mathematical induction.

First, let us formalize our recursion relation. Let $f(p,n)$ be the number of terms in the multinomial expansion for p terms with a power of n. In general, we have it that:

$$\left(x_1+...+x_p\right)^n=\sum_{k=0}^{n}\binom{n}{k}x_1^{n-k}\left(x_2+...+x_p\right)^k$$

If the multinomial expansion for $\left(x_2+...+x_p\right)^k$ has $f(p-1,k)$ terms, then this means the expansion for $\left(x_1+...+x_p\right)^n$ receives a contribution of $f(p-1,k)$ terms at each value of k in the summation, i.e.

$$f(p,n)=\sum_{k=0}^{n}f(p-1,k)\,. \tag{1}$$

We can also repeat the recursion several times to obtain a non-recursive expression:

$$\begin{aligned}f(p,n)&=\sum_{k=0}^{n}f(p-1,k)\\&=\sum_{k=0}^{n}\sum_{j=0}^{k}f(p-2,j)\\&=...\\&=\sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\sum_{n_3=0}^{n_2}...\sum_{n_{p-1}=0}^{n_{p-2}}f(1,n_{p-1})\end{aligned}$$

$$f(p,n)=\sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\sum_{n_3=0}^{n_2}...\sum_{n_{p-1}=0}^{n_{p-2}}1 \tag{2}$$

Where we used that $f(1,n)=1$ for all n, since x^n has exactly one term. Let us study the first few cases $p=1,\ 2,\ 3...$ to hypothesise a general law then prove it inductively.

- ***p* = 1:** As we mentioned earlier, $f(1,n)=1$.

- ***p* = 2:** This is just the binomial expansion. We apply Equation (1): $f(2,n)=\sum_{n_1=0}^{n}1=n+1$

- ***p* = 3:** We again apply Equation (1): $f(3,n)=\sum_{n_1=0}^{n}(n_1+1)=\sum_{k=1}^{n+1}k=\frac{1}{2}(n+1)(n+2)$.

  One may confirm this result via some examples: for *n* = 2, the expansion of $(x+y+z)^2$ $=x^2+y^2+z^2+2xy+2xz+2yz$ has $\frac{1}{2}(2+1)(2+2)=6$ terms; for *n* = 1, $(x+y+z)^1$ has $\frac{1}{2}(1+1)(1+2)=3$ terms, etc.

- ***p* = 4:** Applying Eq (1),

$$
\begin{aligned}
f(4,n)&=\frac{1}{2}\sum_{n_1=0}^{n}(n_1+1)(n_1+2)=\frac{1}{2}\sum_{n_1=0}^{n}\left(n_1^{\,2}+3n_1+2\right)\\
&=\frac{1}{2}\sum_{n_1=0}^{n}n_1^{\,2}+\frac{3}{2}\sum_{n_1=0}^{n}n_1+\sum_{n_1=0}^{n}1\\
&=\frac{1}{6}n\left(n+1/2\right)\left(n+1\right)+\frac{3}{2}\frac{n^2+n}{2}+(n+1)\\
&=\frac{1}{6}(n+1)(n+2)(n+3)
\end{aligned}
$$

  Once again, this may be confirmed via some examples: for *n* = 2, $(x+y+z+w)^2$ equals $x^2+y^2+z^2+w^2+2xy+2xz+2xw+2yz+2yw+2zw$ has $\frac{1}{6}\times3\times4\times5=10$ terms.

- ***p* = 5:** Applying Eq (1),

$$
\begin{aligned}
f(5,n)&=\frac{1}{6}\sum_{n_1=0}^{n}(n+1)(n+2)(n+3)=\frac{1}{6}\sum_{n_1=0}^{n}(n^3+6n^2+11n+6)\\
&=\frac{1}{6}\sum_{n_1=0}^{n}n^3+\sum_{n_1=0}^{n}n^2+\frac{11}{6}\sum_{n_1=0}^{n}n+\sum_{n_1=0}^{n}1\\
&=\frac{1}{6}\left(\frac{n(n+1)}{2}\right)^2+\frac{n(n+1/2)(n+1)}{3}+\frac{11}{6}\frac{n(n+1)}{2}+(n+1)\\
&=\frac{1}{24}(n+1)(n+2)(n+3)(n+4)
\end{aligned}
$$

Based on the five cases above, we conjecture the following:

$$
f(p,n)=\frac{1}{(p-1)!}\prod_{k=1}^{p-1}(n+k)
$$

Where the symbol $\prod_{k=1}^{N} a_k = a_1 a_2 \ldots a_N$. We may also write the above as:

$$f(p,n) = \frac{1}{(p-1)!}\frac{(n+p-1)!}{n!}$$

We may simplify this further with the use of combinatoric coefficients $\binom{N}{R} = \frac{N!}{(N-R)!R!}$:

$$f(p,n) = \binom{n+p-1}{n} \qquad (3)$$

Equation (3) is the conjecture we will attempt to prove through induction for varying values of *p*.

First, note that (3) holds true for the base case $p = 1$, where $f(1,n) = 1 = \binom{n}{n}$, which is indeed what the right-hand side simplifies to.

Next, suppose the conjectured statement holds true for some $p = p_0$. We will now demonstrate that this necessarily implies that the statement is true when $p = p_0 + 1$.

Since the statement is true for $p = p_0$, $f(p_0, n) = \binom{n+p_0-1}{n}$. We again employ Equation (1) to find an expression for $f(p_0+1, n)$:

$$f(p_0+1,n) = \sum_{k=0}^{n} f(p_0,k) = \sum_{k=0}^{n}\binom{k+p_0-1}{k}$$
$$= \binom{p_0-1}{0} + \binom{p_0}{1} + \binom{p_0+1}{2} + \ldots \binom{p_0+n-1}{n}$$

While the proof might seem non-trivial at this point, this is in fact the "hockey-stick pattern" found in Pascal's triangle. The figure below illustrates this pattern:

1
1 1
1 2 1
1 **3** 3 1
1 4 **6** 4 1
1 5 10 **10** 5 1
1 6 15 *20* 15 6 1
1 7 21 35 35 21 7 1

Figure 1: Pascal's triangle -- Hockey-stick property

The numbers indicated in bold in the above diagram add up to the underlined number. This is a general property of Pascal's triangle[1], and can be stated algebraically as follows:

$$\binom{a}{0}+\binom{a+1}{1}+\ldots+\binom{a+h}{h}=\binom{a+h+1}{h}$$

Or more compactly,

$$\sum_{k=0}^{h}\binom{a+k}{k}=\binom{a+h+1}{h} \qquad (4)$$

Substituting $a=p_0-1$, $h=n$, we have:

$$\sum_{k=0}^{n}\binom{p_0+k-1}{k}=\binom{p_0-1+n+1}{n}=\binom{n+(p_0+1)-1}{n}$$

Which tells us that $f(p_0+1,n)=\binom{n+(p_0+1)-1}{n}$, completing our proof (as we know the conjecture is true for $p = 1$, and that it is true for $p=p_0+1$ if it is true for $p=p_0$, hence by induction it is true for all positive integer p).

(We used the hockey-stick pattern (4) as a lemma for our proof, so let us quickly verify, via induction, that it is true. First, observe that (4) is true for $h = 0$, where both sides evaluate to 1. Now suppose it is true for $h=h_0$, so that $\sum_{k=0}^{h_0}\binom{a+k}{k}=\binom{a+h_0+1}{h_0}$. Then

$$\begin{aligned}\sum_{k=0}^{h_0+1}\binom{a+k}{k}&=\sum_{k=0}^{h_0}\binom{a+k}{k}+\binom{a+h_0+1}{h_0+1}\\&=\binom{a+h_0+1}{h_0}+\binom{a+h_0+1}{h_0+1}\\&=\binom{a+h_0+2}{h_0+1}\end{aligned}$$

Completing the proof of our lemma, since we know the conjecture is true for $h = 0$, and that it is true for $h=h_0+1$ if it is true for $h=h_0$, hence by induction it is true for all positive integer h.

The fact that we used induction twice is interesting, and a result of the fact that f is a function of two variables.)

Induction (or perhaps "two-dimensional induction") on the algebraic form of our conjecture is thus a satisfactory approach to proving it. The connection to Pascal's triangle, however, is

[1] Wang, C. e. (2013, November 19). *Pascal's triangle*. Retrieved from Math Forum: http://mathforum.org/mathimages/index.php/Pascal's_triangle#Hockey_Stick_Pattern

intriguing, and encourages us to study it further. Indeed, we will see that we may develop another approach to finding the number of terms, based on Pascal's triangle.

§3. VIA PASCAL'S SIMPLEX

Many properties of binomial coefficients are linked to Pascal's triangle, and it's reasonable to expect a deeper relationship between our research question and this construction beyond just the usage of a property that can be visualized on this triangle.

Observe that the coefficients of a binomial (i.e. $p = 2$) expansion of degree n are just the non-zero (we mention "non-zero" because some constructions of Pascal's triangle involve the triangle surrounded by an infinite number of zeroes to the left and right of each row) terms of the n^{th} row of Pascal's triangle. Therefore, the number of terms in a binomial expansion of degree n is the number of terms on the n^{th} row of Pascal's triangle.

Similarly, if we take the case $p = 1$, the coefficient is always 1, and we can write this down in the form of a 1-dimensional construction. The coefficients on the n^{th} row of this figure represent the coefficients on the expansion of x^n, which is simply x^n, and the fact that there is only 1 term there, which is 1, reflects the fact that that there is only one term in the expansion of x^n, and the coefficient on it is 1.

1
1
1
…

Figure 2: "Pascal's line" -- a 1-dimensional representation of the coefficients on expansions of the form x^n

This lead me to wonder if one may similarly construct a three-dimensional "Pascal's tetrahedron" for the terms of a trinomial expansion, and similarly a "Pascal's simplex" (a simplex is the n-dimensional version of a tetrahedron) for a general multinomial expansion.

It turns out that we can, and the argument for doing so proceeds as following: in the binomial expansion, each term takes the form $x^i y^j$. The exponents on x and y in this term necessarily add up to a fixed n. Hence there is only 1 degree of freedom in choosing the exponents (the other is simply n minus the other exponent), and the terms can be arranged in a line with one exponent increasing and the other decreasing. A row of Pascal's triangle is indeed one-dimensional.

On the other hand, the terms of the trinomial expansion take the form $x^i y^j z^k$, and the three exponents add up to a fixed n. This means there are 2 degrees of freedom. A plot of the different combinations (i.e. different terms in expansion) will in fact be a discrete ternary plot[2] (a two-dimensional, triangular plot of variables with a fixed sum) with $n + 1$ discrete values marked on each side (since any given index can take a value between 0 and n). Placing the coefficients of the trinomial expansion on the lattice points of this ternary plot, one has created the n^{th} layer of Pascal's tetrahedron, and the number of terms in this layer is equal to $f(3, n)$.

[2] Stover, C. (n.d.). *Ternary Diagram*. Retrieved from Wolfram Mathworld: http://mathworld.wolfram.com/TernaryDiagram.html

Similarly, one may construct a "Pascal's simplex" for a general value of p, exploiting the fact that there are $n - 1$ degrees of freedom in choosing the exponents on each term of the expansion.

Our problem has therefore been reduced to finding the number of terms in a given layer of Pascal's simplex of given dimension. Pascal's simplex is documented in mathematical literature[3] – in terms of the number of points, it is indistinguishable from any other discrete simplex. According to existing mathematical literature[4], the total number of points in the first n layers of a discrete r-simplex is $\binom{n+r}{n}$.

This refers to the *total* number of points in the first n layers, not the n^{th} layer of the simplex itself. However, these layers comprise the n^{th} layer of what is the $(r + 1)$-simplex, therefore setting $p = r + 1$, we may say that the number of points on the n^{th} layer of a p-simplex is $\binom{n+p-1}{n}$, thus:

$$f(p,n) = \binom{n+p-1}{n}$$

However, this is not a genuine alternative proof of the theorem at all, as the formula $\binom{n+r}{n}$ is derived based on the very summations discussed in Section 2. Specifically, the number of points on a discrete simplex is typically calculated (e.g. in References 3-4) based on the recursion formula in Equation (1), whose geometric equivalent is essentially the statement "the total number of points in a simplex is equal to the sum of the numbers of points in its layers, which are also simplices of dimension one less than itself and varying size from 0 to n". So our proof in this section could rather be called a *geometric interpretation* of the argument in the first section.

We are inclined to study genuine alternative proofs, and will do so in the following section.

§4. VIA A GENERATING FUNCTION

Ignoring that we've already proved $f(p,n)$ $\binom{n+p-1}{n}$, we will sample a few values of $f(p,n)$:

n* \ *p	**1**	**2**	**3**	**4**	**5**
0	1	1	1	1	1
1	1	2	3	4	5
2	1	3	6	10	15
3	1	4	10	20	35
4	1	5	15	35	70
5	1	6	21	56	126

Table 1: Sample values of f(p, n)

[3] Woods, D., & M.K., K. (1973). Pascal's k-simplex. *The Two-Year College Mathematics Journal*, 38-43.
[4] user109775. *Simple enumeration of discrete simplex*. (2014, September 8). Retrieved from Math Stack Exchange: math.stackexchange.com/q/923187

At some point, however, we stopped actually counting the number of terms in the expansion (it would be foolish to count the 126 terms in a fifth degree 5-nomial expansion) and instead used the following observation:

The number in any cell of Table 1 is the sum of the number directly above it and the number to its left.

Algebraically,

$$f(p,n) = f(p-1,n) + f(p,n-1) \tag{5}$$

This identity is intuitively clear – for instance, one may take

$$(x+y+z+w)^3 = (x+y+z)^3 + 3(x+y+z)^2 w + 3(x+y+z)w^2 + w^3$$

And notice that $3(x+y+z)^2 w + 3(x+y+z)w^2 + w^3$ necessarily takes the same number of terms as the expansion of $(x+y+z+w)^2 = (x+y+z)^2 + 2w(x+y+z) + w^2$. In our "simplex" interpretation, this is equivalent to splitting the last layer off the simplex. We prove the result more conclusively as follows:

Notice that any cell in Table 1 equals the sum of the numbers in all the cells of the column directly to the left of it whose *n*-parameters are not greater than its own. This is simply a restatement of the recurrence relation in (1):

$$f(p,n) = \sum_{k=0}^{n} f(p-1,k)$$

For instance, $f(4, 4)$ is the sum of all $f(3, k)$ for k between 0 and 4 inclusive. However, at the same time, the cell $f(4, 3)$ is the sum of all $f(3, k)$ for k between 0 and 3 inclusive!

$$35 = 1+3+6+10+15$$
$$20 = 1+3+6+10$$

Hence the difference between them is simply $f(3, 4)$, thus $f(4,4) = f(4,3) + f(3,4)$. Our general proof is simply:

$$\begin{aligned} f(p,n) &= \sum_{k=0}^{n} f(p-1,k) \\ &= f(p-1,n) + \sum_{k=0}^{n-1} f(p-1,k) \\ &= f(p-1,n) + f(p,n-1) \end{aligned}$$

Which conclusively proves Equation (5). The symmetry between p and n in this relation is intriguing, as the roles they satisfy in the multinomial expansion are quite different.

This recurrence relation (5), even though two-dimensional, is advantageous over Equation (1) for several reasons:

- As it recurs in both p and n, it doesn't require knowledge of how the function behaves over either "axis" (i.e. parameter), simply some initial conditions.
- As it does not involve the usage of a summation with a varying limit, there are a multitude of techniques available to solve such a recursion relation, such as the usage of a generating function[5].

The "generating function" of an infinite sequence is a function defined to have a Taylor series whose coefficients are the terms of the infinite sequence. If we can find the closed form of this function through the recurrence relation alone, we can extract the coefficients $f(p, n)$ from its Taylor series.

Generating functions are typically defined for one-dimensional recurrence relations, e.g. $g(n) = g(n-1) + g(n-2)$ (the Fibonacci relation), for which the standard Taylor series suffices. However, as we are dealing with a two-dimensional recurrence relation (i.e. we have two variables, p and n), I was unable to find a standard procedure to derive the generating function, so decided to develop my own procedure based on the standard one for one-dimensional recurrence.

First of all, I used the multivariable Taylor series with two variables. Equation (6) shows the 2-variable Taylor series[6] centred around (0, 0):

$$\begin{aligned} Z(x,y) &= Z(0,0) \\ &+\frac{1}{1!}\left[x\,Z_x(0,0)+y\,Z_y(0,0)\right] \\ &+\frac{1}{2!}\left[x^2\,Z_{xx}(0,0)+2xy\,Z_{xy}(0,0)+y^2\,Z_{yy}(0,0)\right] \\ &+\frac{1}{3!}\left[x^3\,Z_{xxx}(0,0)+3x^2y\,Z_{xxy}(0,0)+3xy^2\,Z_{xyy}(0,0)+y^3\,Z_{yyy}(0,0)\right] \\ &+\ldots \end{aligned} \tag{6}$$

We define a generating function

$$Z(x,y) = \sum_{n,p=0}^{\infty} f(p+1,n)x^p y^n$$

(We use $f(p+1,n)$ because f is not defined for $p = 0$). for some arbitrarily introduced variables x and y. Then (6) implies that

$$f(p+1,n) = \frac{1}{(n+p)!}\binom{n+p}{p}\partial_x^p \partial_y^n Z(0,0)$$

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[5] NPTEL Lectures (2014). Selected Topics in Mathematical Physics - Calculus of Residues, Part II [Recorded by V. Balakrishnan]. IIT Madras.

[6] Sullivan, E. (2014). Taylor Series: Single Variable and Multi-Variable. Retrieved from Eric Sullivan's Homepage, Website: http://www.math.ucdenver.edu/~esulliva/Calculus3/Taylor.pdf

$$f(p+1,n)=\frac{\partial_x^p\partial_y^n Z(0,0)}{p!n!} \tag{7}$$

Thus we turn our attention towards finding $Z(x,y)$ so we can find an expression for its repeated partial derivatives, evaluate them at zero and substitute them into (7).

To do so, we first rewrite (5) as:

$$f(p+2,n+1)=f(p+1,n+1)+f(p+2,n) \tag{8}$$

Which is the same statement with a variable substitution $p\to p+2,\ n\to n+1$. This is to ensure that in the terms of our summation, *p* is always positive and *n* is always non-negative.

Next, we multiply (8) by $x^p y^n$ and sum over *p* and *n*, both variables ranging from 0 to infinity.

$$\sum_{n,p=0}^{\infty} x^p y^n f(p+2,n+1)=\sum_{n,p=0}^{\infty} x^p y^n f(p+1,n+1)+\sum_{n,p=0}^{\infty} x^p y^n f(p+2,n)$$

We want to rewrite each of these terms as an expression in terms of *f*(*z*) so as to obtain an algebraic expression for *f*(*z*). We do this in the following way:

$$\frac{1}{xy}\sum_{n,p=0}^{\infty} x^{p+1} y^{n+1} f(p+2,n+1)=\frac{1}{y}\sum_{n,p=0}^{\infty} x^p y^{n+1} f(p+1,n+1)+\frac{1}{x}\sum_{n,p=0}^{\infty} x^{p+1} y^n f(p+2,n)$$

$$\frac{1}{xy}\sum_{p=1,n=1}^{\infty} x^p y^n f(p+1,n)=\frac{1}{y}\sum_{p=0,n=1}^{\infty} x^p y^n f(p+1,n)+\frac{1}{x}\sum_{p=1,n=0}^{\infty} x^p y^n f(p+1,n)$$

Where we transformed the index $p\to p-1$ so that e.g. $p=0$ becomes $p-1=0$. We now split each summation into *Z*(*x*, *y*) and some remainder terms:

$$\frac{1}{xy}\left[Z(x,y)-\sum_{n=0}^{\infty} y^n f(1,n)-\sum_{p=0}^{\infty} x^p f(p+1,0)+f(1,0)\right]$$
$$=\frac{1}{y}\left[Z(x,y)-\sum_{p=0}^{\infty} x^p f(p+1,0)\right]+\frac{1}{x}\left[Z(x,y)-\sum_{n=0}^{\infty} y^n f(1,n)\right]$$

We know that $f(1,n)=1$ (since a 1-nomial expansion has only 1 term), that $f(p+1,0)=1$ (since anything raised to the power of 0 is 1, which has only 1 term) and that $f(1,0)=1$. So we may further simplify as:

$$\frac{1}{xy}\left[Z(x,y)-\sum_{n=0}^{\infty} y^n-\sum_{p=0}^{\infty} x^p+1\right]=\frac{1}{y}\left[Z(x,y)-\sum_{p=0}^{\infty} x^p\right]+\frac{1}{x}\left[Z(x,y)-\sum_{n=0}^{\infty} y^n\right]$$

And:

$$\frac{1}{xy}\left[Z(x,y)-\frac{1}{1-y}-\frac{1}{1-x}+1\right]=\frac{1}{y}\left[Z(x,y)-\frac{1}{1-x}\right]+\frac{1}{x}\left[Z(x,y)-\frac{1}{1-y}\right]$$

Where we evaluated the sum of an infinite geometric series. However, as these geometric series only converge for $|x|<1$ and $|y|<1$ correspondingly, we must restrict the domain of our generating function to these radii of convergence for each variable.

We may now simplify algebraically for $Z(x,y)$:

$$(1-x-y)Z(x,y)=\frac{1}{1-y}+\frac{1}{1-x}-\frac{x}{1-x}-\frac{y}{1-y}-1$$
$$(1-x-y)Z(x,y)=1$$

Which gives us our grand result:

$$Z(x,y)=\frac{1}{1-x-y} \tag{9}$$

Where $|x|<1$ and $|y|<1$.

Recall Equation (7) – in order to use it, we must find the general partial derivative $\partial_x^p \partial_y^n Z(x,y)$. We start by listing the derivatives of *Z* with respect to *x*.

$$\partial_x^0 Z=\frac{1}{1-x-y}$$
$$\partial_x^1 Z=\frac{1}{(1-x-y)^2}$$
$$\partial_x^2 Z=\frac{2!}{(1-x-y)^3}$$
$$\partial_x^3 Z=\frac{3!}{(1-x-y)^4}$$

We guess that $\partial_x^p Z=\frac{p!}{(1-x-y)^{p+1}}$, and quickly prove this by induction – the base case $p = 0$ is already proven, so suppose the statement is true for some $p = t$. Then

$$\partial_x^{t+1} Z=\frac{\partial}{\partial x}\frac{t!}{(1-x-y)^{t+1}}=-(-1)\frac{(t+1)t!}{(1-x-y)^{t+2}}=\frac{(t+1)!}{(1-x-y)^{t+2}}$$

Implying that the statement is true for $p = t + 1$. Since it is true for $p = 0$, this means by induction that it is true for all *p*.

Now we repeatedly differentiate $\partial_x^p Z=\frac{p!}{(1-x-y)^{p+1}}$ with respect to *y*.

$$\partial_y \partial_x^p Z=\frac{(p+1)!}{(1-x-y)^{p+2}}$$

$$\partial_y^2 \partial_x^p Z = \frac{(p+2)!}{(1-x-y)^{p+3}}$$

$$\partial_y^3 \partial_x^p Z = \frac{(p+3)!}{(1-x-y)^{p+4}}$$

We guess that $\partial_y^n \partial_x^p Z(x,y) = \frac{(n+p)!}{(1-x-y)^{n+p+1}}$. Again, we quickly prove this by induction – the base case $n = 0$ is already proven above, so suppose it is true for some $n = u$. Then

$$\partial_y^{u+1} \partial_x^p Z(x,y) = \frac{\partial}{\partial y} \frac{(u+p)!}{(1-x-y)^{u+p+1}} = -(-1)\frac{(u+p+1)(u+p)!}{(1-x-y)^{u+p+2}} = \frac{(u+p+1)!}{(1-x-y)^{u+p+2}}$$

Implying that the statement is true for $n = u + 1$. Since it is true for $n = 0$, this means by induction that it is true for all n. Thus we have it that

$$\partial_y^n \partial_x^p Z(x,y) = \frac{(n+p)!}{(1-x-y)^{n+p+1}} \tag{10}$$

Evaluating this at (0, 0), as Equation (7) demands, yields $\partial_y^n \partial_x^p Z(0,0) = (n+p)!$. Therefore:

$$f(p+1,n) = \frac{(n+p)!}{p!n!} \Rightarrow f(p,n) = \frac{(n+p-1)!}{(p-1)!n!}$$

Or simply

$$f(p,n) = \binom{n+p-1}{n}$$

Which is our result, proven using a genuinely independent method.

§5. VIA A COMBINATORIC ARGUMENT

I would like to go back and make an observation about our reasoning in Section 4. To count the number of points in a discrete simplex, we simply said that it can be done via induction. However, we may alternatively generalize the definition of the ternary plot and say that the number of points in a simplex is simply the number of p-tuples of numbers whose sum is n.

In fact, this can be inferred directly from the form taken by the multinomial expansion, where we have that the sum of the p indices in each term adds up to n and since all such combinations are present in the expansion, $f(p, n)$ is simply the number of ways to distribute n identical objects into p distinct boxes. We may solve this with the "stars-and-bars method", where we consider the boundaries of the boxes as $p - 1$ "bars" and place them alongside the n objects ("stars"). We then look at the number of ways to place thee bars among the $n + p - 1$ places, which is simply $\binom{n+p-1}{n}$. This is a slightly simpler proof than that discussed in Section 4.

§6. CONCLUSION

We have explored a variety of ways to find the number of terms in a multinomial expansion, and also described a geometric interpretation (that as we saw in Section 5, can be extended not only to the inductive proof) for the same. Seeing the relatively complicated calculation simplify to our desired expression was satisfying, and a testimony to mathematical elegance.

The function we derived (repeatedly) is unfortunately only applicable to positive values of n – for negative values of n, it either returns 0 or undefined (in the latter case, the limit approaches 1) if evaluated using the gamma function, whereas the actual expansion (through Taylor series, such as its special case in Newton's binomial theorem) has an infinite number of terms. This limitation produces an opportunity, however, of discovering regularization techniques (similar to the famous Ramanujan summation).

The $f(p, n)$ table (Table 1) is interesting in its own right, and is worth further study. For example, I observed that the weighted sum (where the weights are binomial coefficients) of elements in the diagonals of right-angled triangles centered at a point are zero.

n / ***p***	**1**	**2**	**3**	**4**	**5**
0	1	1	1	1	1
1	1	2	3	4	5
2	1	3	6	10	15
3	1	4	10	20	35
4	1	5	15	35	70
5	1	6	21	56	126

Table 2: Illustration of right-angled triangle property

I.e. if one weights the values in the green cells by 1, 4, 6, 4 and 1 starting from either end of the line segment, then the weighted sum of these numbers is 70. Algebraically in general,

$$f(n,p)=\sum_{i=0}^{k}\binom{k}{i}f(n-k+i,p)$$

Investigating the proof of this will be a topic of future research. Hence, the exploration was not only a mathematically appealing exercise, but also shows scope for future research and theoretical application.

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