The generalised Cauchy derivative as a principle value of the Grünwald-Letnikov derivative for divergent expansions

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GRÜNWALD-LETNIKOV

$$f''(x) = \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

$$\vdots$$

$$f^{(n)}(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f\big(x + (n-k)h\big)$$

(Resemblance to binomial theorem is suggestive)

GRÜNWALD-LETNIKOV

$$\phi^k f(x) := f(x + kh)$$

- ϕ^k forms a "commutative ring" – so we have the binomial theorem
- Motivation: $\phi^1 - \phi^0 = d$ (the differential operator)
- Consider $(\phi^1 - \phi^0)^n$
- Like with binomial theorem, extend this to non-integer *n*

$$f^{(R)}(x) = \lim_{h\to 0} \frac{1}{h^R} \sum_{k=0}^{\infty} \binom{R}{k} (-1)^k f(x + (R-k)h)$$

RIEMANN-LIOUVILLE

- Or "Generalised Cauchy" – as it follows from Cauchy's Repeated Integral formula
- Straightforward form for power functions

$$\frac{d^n}{dx^n} x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

- **Proof by Ortigueira & Coito (2004):** Riemann-Liouville derivative equals the Grünwald-Letnikov derivative

COUNTER-EXAMPLE

- Grünwald-Letnikov – divergent

$$D^{1/2}x = \lim_{h\to 0}\left[xh^{-1/2}\sum_{k=0}^{\infty}\binom{1/2}{k}(-1)^k - h^{1/2}\sum_{k=0}^{\infty}\binom{1/2}{k}(-1)^k k\right]$$

$$= \lim_{h\to 0}\left[\frac{\sqrt{h}}{2}\sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k\right]$$

- Riemann-Liouville

$$D^{1/2}x = \frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2} = 2\sqrt{x/\pi}$$

IDEA

- Divergence of the form $\sqrt{h/0}$ – i.e. summation diverges "too fast"
- There are two limits here, $h \to 0$ and $N \to \infty$
- **New approach:** take the two limits "together" (i.e. writing h as a function of N)
  - Potentially infinite possible values to the limit depending on which function you choose
  - Idea: let the "principal value" be the value for which the derivative coincides with RL
- **Question:** What function achieves this principal value?

STANDARD RESULTS AND LEMMAS

- (Lemma 1) $$\sum_{k=0}^{N} \binom{R}{k} (-1)^k = (-1)^N \binom{R-1}{N}$$
- (Lemma 2) $$(-1)^N \binom{R}{N} = -\frac{\sin(\pi R)}{\pi} \Gamma(R+1) N^{-R-1} + O(N^{-R-2})$$
- (Lemma 3) $$\sum_{k=0}^{N} \binom{R}{k} (-1)^k k^j = (-1)^N \frac{R}{R-j} \binom{R-1}{N} N^j + O(N^{j-1})$$

EXAMPLE: $D^{1/2}x$

$$D^{1/2}x = \lim_{h\to 0, N\to\infty} \frac{x}{\sqrt{h}} \sum_{k=0}^{N} \binom{1/2}{k} (-1)^k - \sqrt{h} \sum_{k=0}^{N} k \binom{1/2}{k} (-1)^k$$

- Lemma 1, 3: $$D^{1/2}x = \frac{x}{\sqrt{h}} (-1)^N \binom{-1/2}{N} + \frac{\sqrt{h}}{2} (-1)^N \binom{-3/2}{N}$$

- Lemma 2: $$(-1)^N \binom{-1/2}{N} \sim \frac{1}{\sqrt{\pi}} N^{-1/2};\ (-1)^N \binom{-3/2}{N} \sim \frac{2}{\sqrt{\pi}} N^{1/2}$$

$$D^{1/2}x = \frac{x}{\sqrt{h}} \frac{1}{\sqrt{\pi}} N^{-1/2} + \frac{\sqrt{h}}{2} \frac{2}{\sqrt{\pi}} N^{1/2}$$

- Solve for h: $$h = x/N$$

MORE Q-RELATIONS

- For all derivatives of power functions, $\exists q, h = q\,x/N$.

$$q - 2\sqrt{q} + 1 = 0 \qquad (D^{1/2}x)$$
$$q^R - Rq + (R-1) = 0 \qquad (D^R x)$$
$$q^{-R} \quad {}_2F_1(-m, -R; 1-R; q) = \frac{\pi R}{\sin(\pi R)}\binom{m}{R} \qquad (D^R x^m)$$

- Special cases
  - $q = 1$ is always a solution – this means it is a solution for all analytic functions
  - Integer R – allows $q = 0$ (or really any value of q)
  - $R = 1/3, m = 1$ – allows $q = -1/8$
  - $m = 0$ allows q satisfying $q = e^{k \cdot 2\pi i/R}$ for $k \in \mathbb{Z}$

(IMPLICATIONS)

HANDEDNESS OF GL DERIVATIVE

- How do you define ordering of terms in a binomial series? For the binomial series on real numbers, it doesn't matter
  - Can be proven: different orderings equivalent to different directions of taking the limit
  - So if the limit *exists*, the ordering doesn't matter
- But we're dealing with instances where it doesn't exist!
- Sign/argument of q tells us the correct handedness

$$f^{(R)}(x) = \lim_{h \to 0} \sum_{k=0}^{\infty} \binom{R}{k} (-1)^{R-k} f(x + kh)$$

(IMPLICATIONS)

DISCONTINUOUS FUNCTIONS

- Fractional derivative inherently “non-local” – can see this from infinite summation form of GL derivative, or integral form of RL
- Can the fractional derivative “sense” discontinuities/non-smoothness in the function?
- Can be conclusively answered with my principal value formalism! – “Yes, to a certain extent.”
  - Where the “extent” equals qx

FUTURE RESEARCH

- Other (than 1) values of q for a general power series
- General theory for fractional derivatives of generally non-smooth functions
- Alternative formalization of Dirac Delta function!
  - Instead of taking limits of functions, take limit $R \to 1$ for $D^R H(x)$

REFERENCES

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[3] V. Kiryakova, A long standing conjecture failed? In: Proceedings of the 2nd Int'l Workshop "Transform Methods & Special Functions, Varna '96", IMI-BAS, Sofia (1996), 579-588.

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APPENDIX

- Cauchy's Repeated Integral formula

$$f^{(-n)}(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\, dt$$

- Explicit restatement of q-relation for $D^R x^m$:

$$q^{-R}\sum_{j=0}^{m}\binom{m}{j}\frac{(-q)^j}{R-j} = \frac{\pi}{\sin\pi R}\binom{m}{R}$$