The σ-algebra inverse limit thing

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13 November 2022

1 Dummy

Definition 1.1 (Maximal Introspection – single-observer)**.** Let $\mathbf{F}^0$ be a set, called the base partition, and define a sequence $(\mathbf{F}^n)$ by the following recurrence:

$$\mathbb{P}^n = \left\{ \mathbf{P} : \mathbf{F}^n \to [0,1] \;\middle|\; \sum_{f \in \mathbf{F}^n} \mathbf{P}(f) = 1 \right\}$$
$$\mathbf{F}^{n+1} = \mathbf{F}^0 \times \mathbb{P}^n$$

($\mathbb{P}^n$ should be read as the probability measure defined by "the $(n+1)$-level agent on the n-level partition") – and define for $m \leq n$ the projection $\pi^{mn} : \mathbf{F}^n \to \mathbf{F}^m$ transitively via composition on the following recurrence:

$$\pi^{01}\left(f^0, \mathbf{P}^0\right) = f^0$$
$$\pi^{m(m+1)}\left(f^0, \mathbf{P}^m\right) = \left\langle f^0,\ \left(g \in \mathbf{F}^{m-1}\right) \mapsto \sum_{h \in \pi^{(m-1)m^{-1}}(g)} \mathbf{P}^m(h) \right\rangle$$

Then $(\mathbf{F}^n)$ forms an inverse family under these connecting morphisms, and we call the inverse limit $\mathbf{F} := \varprojlim \mathbf{F}^n$ the "maximally introspective extension" of $\mathbf{F}^0$.

Concretely, an element of $\mathbf{F}$ is an element of $\mathbf{F}^0$ along with a sequence of probability distributions such that each successive probability distribution that "refines" the previous. Examples of "interpreting"/coarsening an event in $\mathbf{F}^{n+1}$ as an event in $\mathbf{F}^n$ (because this is the tricky bit) in Figs 1, 2, 3 respectively.

To be sure, this definition is pretty lacking – it doesn't quite suffice for formulating Gödel's theorem, because any program you place in $\mathbf{F}^0$ is unable to refer to the agent's beliefs, which are located in the finer σ-algebras. A fuller description – for an agent rather than an observer, and for multiple agents – would require a more general (perhaps category-theoretic) definition, which would also provide a formal justification for this inverse-limit construction. I can probably do this fairly easily.

$$\pi^{01} \quad : \quad \begin{array}{l} 0 \left.\vphantom{X}\right\} \ {\color{red}0.5} \\[1em] 1 \left.\vphantom{X}\right\} \ {\color{red}0.5} \end{array} \quad 0 \quad \mapsto \quad 0$$

Figure 1: π^{01}

$$\pi^{12} \quad : \quad \begin{array}{l}
\begin{array}{l} 0 \left.\vphantom{X}\right\} \ {\color{red}0.5} \\ 1 \left.\vphantom{X}\right\} \ {\color{red}0.5} \end{array} \ 0 \left.\vphantom{\begin{array}{l}X\\X\end{array}}\right\} \ {\color{red}0.3} \\[1em]
\begin{array}{l} 0 \left.\vphantom{X}\right\} \ {\color{red}0.5} \\ 1 \left.\vphantom{X}\right\} \ {\color{red}0.5} \end{array} \ 1 \left.\vphantom{\begin{array}{l}X\\X\end{array}}\right\} \ {\color{red}0.4} \\[1em]
\begin{array}{l} 0 \left.\vphantom{X}\right\} \ {\color{red}0.1} \\ 1 \left.\vphantom{X}\right\} \ {\color{red}0.9} \end{array} \ 0 \left.\vphantom{\begin{array}{l}X\\X\end{array}}\right\} \ {\color{red}0.2} \\[1em]
\begin{array}{l} 0 \left.\vphantom{X}\right\} \ {\color{red}0.1} \\ 1 \left.\vphantom{X}\right\} \ {\color{red}0.9} \end{array} \ 1 \left.\vphantom{\begin{array}{l}X\\X\end{array}}\right\} \ {\color{red}0.1}
\end{array} \quad 0 \quad \mapsto \quad \begin{array}{l} 0 \left.\vphantom{X}\right\} \ {\color{red}0.5} \\[1em] 1 \left.\vphantom{X}\right\} \ {\color{red}0.5} \end{array} \quad 0$$

Figure 2: π^{12}

But this does provide us with a toy model to figure out exactly how an inverse-limit construction avoids Cantorian issues, whether this $\mathbf{F}$ should be interpreted as a σ-algebra or a sample space, etc.

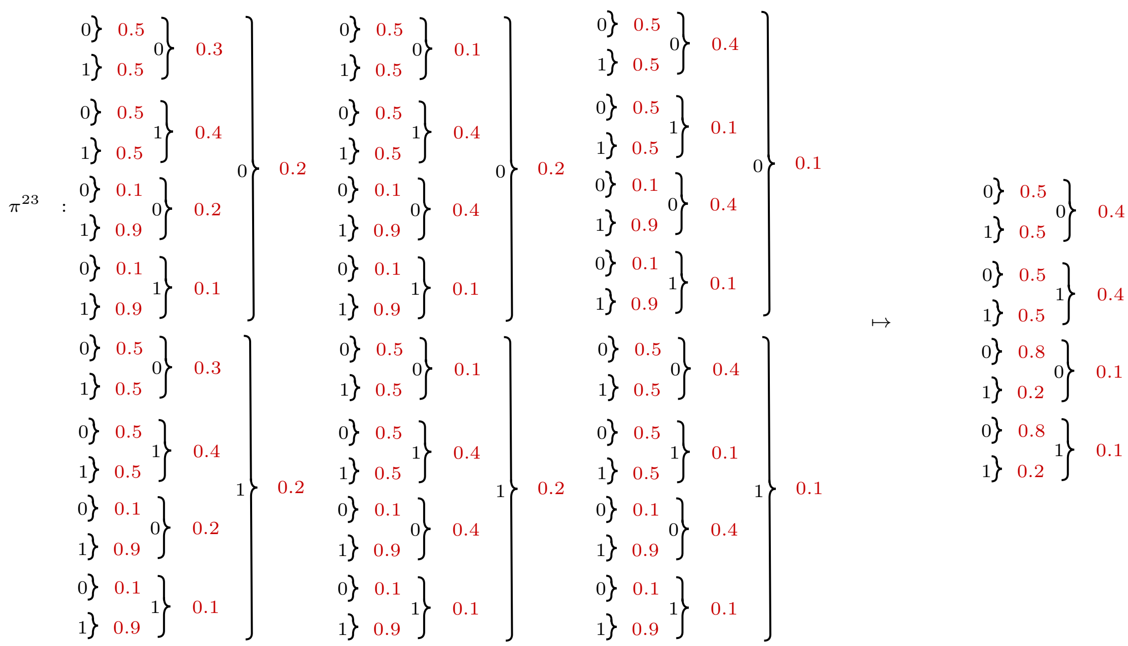

Figure 3: π^{23}