Max-Sum and other things

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1 Max-sum algorithm

1.1 Definitions and intuition

The setting of the max-sum algorithm is a bipartite graph between choices x_i's and utility function U_j's (e.g. Fig 1.1). Each x_i takes values in a discrete set X_i, and each U is a function on $\prod_i X_i$:

Each iteration, a "message is passed from each U_j to each of its neighbours X_i", that is a function is calculated, $r_{ji} : X_i \to \mathbb{R}$ defined as:

$$r_{ji}(x_i) = \max_{x_{-i}} \left[U_j(x_i, x_{-i}) + \sum_{i' \neq i} q_{i'j}(x_{i'}) \right] \tag{1}$$

And a "message is passed from each X_i to each of its neighbours U_j", that is a function is calculated, $q_{ij} : X_i \to \mathbb{R}$ defined as:

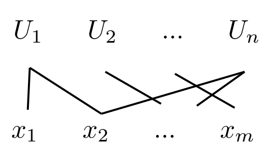
$$q_{ij}(x_i) = \sum_{j' \neq j} r_{j'i}(x_i) \tag{2}$$

Each x_i^* is set to maximize $\sum_j r_{ji}(x_i)$.

When the graph is cycle-free, this algorithm *globally* maximizes the "aggregate utility" $\sum_j U_j(x^*)$ [3].

1.2 Link to markets

Eq 1 essentially allows each agent i whose actions affect stakeholder j to take into account the demand from stakeholder j, $U_j(x_i, x_{-i})$, and the derived demand from i', $q_{i'j}(x_{i'})$. Eq 2 shows the calculation of this derived demand as the sum of demands from other agents.



$r_{ji}(x_i)$ represents the payment received by agent i from stakeholder j. Thus the algorithm is analogous to price propagation in a certain way.

Possible research question 1. The notion of "aggregate utility" is not a fundamental one, as it is not invariant under a reparameterization of the utility function. Here, utility is defined as *value*, the maximum amount of money that a stakeholder would be willing to give to secure a particular choice by some agent. Alternatively, one may wish to maximize a certain "inequality-adjusted" measure of welfare, or some measure that reflects a particular power distribution in society.

I believe it is possible to draw a correspondence between measures of aggregate utility and *rights structures* (defined mathematically e.g. in [1] and my own previous work [2]) – in general, we may ask: *given a particular rights structure or initial endowment, can we find the corresponding functions U_j – and thus the corresponding max-sum algorithm – that reflects the dynamics of the resulting game?*

Possible research question 2. Max-sum in this classical form *assumes* the existence of a standard measure of value (e.g. money). One may instead consider generalizations to barter economies, or where money is modeled as a good like any other. More generally, one may consider *directly dealing with (ordinal) utility functions* rather than the value functions U_j, and discover the value functions from a rights structure (thus tying back to Q1).

1.3 Links to backpropagation?

Given a balanced bipartite graph (so there is a natural correspondence between x_i's and $U_i's$, grouping them as *agents*), we may create a corresponding directed graph $\{\alpha_1, \dots \alpha_n\}$, with an arrow $\alpha_i \to \alpha_j$ iff there's an edge $x_i - U_j$.

Possible research question 3. Note that the max-sum algorithm as we discussed it assumed discrete choice sets X_i – this abstracted away the problem of actually maximizing on this domain. What if we don't, and instead adopted a marginal approach? I there a "differentiable version of max-sum", and could this be related to backpropagation on the corresponding directed graph?

Possible research question 4. If it is, then this raises the question of graphs with cyclicities – in general, there are no theoretical guarantees of convergence in this case, but convergence has been observed in numerous special cases [4, 5, 6]. Counter-examples could be generated motivated by economics, such as from network effects, and more general results could be drawn on the conditions under which non-acyclic computation may be efficient.

References

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